

HOLE PROBABILITIES OF $SU(m+1)$ GAUSSIAN RANDOM POLYNOMIALS

JUNYAN ZHU

ABSTRACT. In this paper, we study hole probabilities $P_{0,m}(r, N)$ of $SU(m+1)$ Gaussian random polynomials of degree N over a polydisc $(D(0, r))^m$. When $r \geq 1$, we find asymptotic formulas and decay rate of $\log P_{0,m}(r, N)$. In dimension one, we also consider hole probabilities over some general open sets and compute asymptotic formulas for the generalized hole probabilities $P_{k,1}(r, N)$ over a disc $D(0, r)$.

0. INTRODUCTION

Hole probability is the probability that some random field never vanishes over some set. The case of Gaussian random entire functions was studied by Sodin and Tsirelson:

Theorem (Sodin, Tsirelson[7] Theorem 1). *Let $\psi(z) = \sum_{k=0}^{\infty} c_k \frac{z^k}{\sqrt{k!}}$, where $c_k (k \geq 0)$ are i.i.d. standard complex Gaussian random variables. Then $\exists C_1 \geq C_2 > 0$ such that*

$$\exp\{-C_1 r^4\} \leq \text{Prob}\{0 \notin \psi(D(0, r))\} \leq \exp\{-C_2 r^4\}.$$

In [9], the authors considered the case of Gaussian random sections: let M be a compact Kähler manifold with complex dimension m and $(L, h) \rightarrow M$ be a positive holomorphic line bundle. γ_N denotes the Gaussian probability measure on $H^0(M, L^N)$ induced by the fiberwisely inner product h^N and the polarized volume form $dV_M = \frac{\omega_h^m}{m!} = \frac{1}{m!} (\frac{\sqrt{-1}}{2\pi} \Theta_h)^m$, where Θ_h is the Chern curvature tensor of (L, h) .

Theorem (Shiffman, Zelditch, Zrebiec[9] Theorem 1.4). *For any nonempty open set $U \subset M$, if there exists $s \in H^0(M, L)$ such that s does not vanish on \bar{U} . Then $\exists C_1 \geq C_2 > 0$ such that for $N \gg 1$,*

$$\exp\{-C_1 N^{m+1}\} \leq \gamma_N\{s_N \in H^0(M, L^N) : 0 \notin s_N(U)\} \leq \exp\{-C_2 N^{m+1}\}.$$

Therefore, it is natural to ask: can we find sharp constants C_1, C_2 in the above two theorems and furthermore, is it possible to obtain an asymptotic formula and a decay rate for the hole probability? Using Cauchy's integral estimates, Nishry answered this question in the random entire function case:

Theorem (Nishry[4] Theorem 1). *Let $\psi(z) = \sum_{k=0}^{\infty} c_k \frac{z^k}{\sqrt{k!}}$, where $c_k (k \geq 0)$ are i.i.d. standard complex Gaussian random variables. Then*

$$\text{Prob}\{0 \notin \psi(D(0, r))\} = \exp\left\{-\frac{e^2}{2} r^4 + O(r^{\frac{18}{5}})\right\}.$$

This inspires us that for those line bundles with polynomial sections, maybe it is possible to find an asymptotic formula for the hole probability.

If $P_{0,m}(r, N)$ denotes the hole probability of $SU(m+1)$ Gaussian random polynomials over the polydisc $(D(0, r))^m$, $d_m x$ is the Lebesgue measure on \mathbb{R}^m and

$$E_r(x) := 2 \sum_{i=1}^m x_i \log r - \left[\sum_{i=1}^m x_i \log x_i + (1 - \sum_{i=1}^m x_i) \log (1 - \sum_{i=1}^m x_i) \right]$$

is a continuous function defined over the standard simplex $\Sigma_m := \{x = (x_1, \dots, x_m) \in \mathbb{R}^{m+} : \sum_{i=1}^m x_i \leq 1\}$ (here we adopt the convention that $0 \log 0 = 0$), we have the following results:

Theorem 0.1. For $r \geq 1$,

$$\log P_{0,m}(r, N) = -N^{m+1} \int_{\Sigma_m} E_r(x) d_m x + o(N^{m+1}),$$

where

$$\int_{\Sigma_m} E_r(x) d_m x = \frac{2m \log r}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k}.$$

Theorem 0.2. For $r > 0$,

$$\begin{aligned} \log P_{0,m}(r, N) &\geq -N^{m+1} \int_{x \in \Sigma_m: E_r(x) \geq 0} E_r(x) d_m x + o(N^{m+1}), \\ \log P_{0,m}(r, N) &\leq -N^{m+1} \int_{x \in \mathbb{R}^{m+1}: \sum_{i=1}^m x_i \leq \alpha_0} E_r(x) d_m x + o(N^{m+1}), \end{aligned}$$

where

$$\alpha_0 = \alpha_0(r, m) = \begin{cases} 1 & \text{if } 2 \log r + \sum_{k=2}^m \frac{1}{k} \geq 0, \\ \text{the nonzero root of } (2 \log r + \sum_{k=2}^m \frac{1}{k})\alpha = \alpha \log \alpha + (1 - \alpha) \log(1 - \alpha) & \text{if } 2 \log r + \sum_{k=2}^m \frac{1}{k} < 0. \end{cases}$$

Here when $m = 1$, we take $\sum_{k=2}^m \frac{1}{k} = 0$.

Remark 0.3. Theorem 0.1 can be derived from Theorem 0.2 as when $r \geq 1$, $\{x \in \Sigma_m : E_r(x) \geq 0\} = \Sigma_m$ and $\alpha_0(r, m) = 1$. In fact we could have proved this general case directly. But the idea of the proof would turn out to be extremely difficult to follow.

Corollary 0.4. In the case of $m = 1$, the asymptotic formula for the logarithm of the hole probability over a disc exists for all $r > 0$:

$$\log P_{0,1}(r, N) = -N^2 \int_0^{\alpha_0} E_r(x) dx + o(N^2),$$

here

$$\int_0^{\alpha_0} E_r(x) dx = \frac{1}{2} \alpha_0 (2 \log r + 1 - \log \alpha_0),$$

and $\alpha_0 = \alpha_0(r, 1) \in (0, 1]$ is given in Theorem 0.2.

Because of the simplicity of one dimensional case, we can obtain more about the hole probability of $SU(2)$ Gaussian random polynomials:

Theorem 0.5. If $U \subset \mathbb{C}$ is a bounded simply connected domain containing 0 and ∂U is a Jordan curve. Let $\phi : D(0, 1) \rightarrow U$ be a biholomorphism given by the Riemann mapping theorem such that $\phi(0) = 0$ (thus ϕ is unique up to the composition of a unitary transformation of \mathbb{C}). Then the hole probability $P_{0,1}(U, N)$ of $SU(2)$ Gaussian random polynomials of degree N over U satisfies

$$\log P_{0,1}(U, N) \leq -(\log |\phi'(0)| + \frac{1}{2}) N^2 + o(N^2).$$

Also in dimension one, it makes sense to study the number of zeros in some set. So let a generalized hole probability $P_{k,1}(r, N)$ be the probability that an $SU(2)$ Gaussian random polynomial of degree N has no more than k zeros in $D(0, r)$, then the following theorem shows that asymptotic formula of $\log P_{k,1}(r, N)$ exists:

Theorem 0.6. For all $k \geq 0$ and $r > 0$:

$$\log P_{k,1}(r, N) = -\frac{1}{2} \alpha_0 (2 \log r + 1 - \log \alpha_0) N^2 + o(N^2),$$

where $\alpha_0 = \alpha_0(r, 1) \in (0, 1]$ is given in Theorem 0.2.

We should remark here that in all the cases we consider, the event that some Gaussian random polynomial has zeros on the boundary of some open set is a null set, i.e. of zero probability. Therefore we do not distinguish between the (generalized) hole probability over an open set and that over its closure.

Acknowledgement. The author would like to express his sincere gratitude to Professor Bernard Shiffman for his patience, motivation, enthusiasm and enlightening discussions. Without his guidance and persistent help this paper would not have been possible.

1. BACKGROUND

We review in this section some background on $SU(m+1)$ Gaussian random polynomials and the definition of our probability measures. Before that, let's define two lexicographically ordered sets that will be consistently used as index sets throughout this paper.

Definition 1.1.

$$\begin{aligned}\Gamma_{m,N} &:= \{J = (j_1, \dots, j_m) \in [0, N]^m \cap \mathbb{Z}^m : 0 \leq j_1 \leq \dots \leq j_m \leq N\}, \\ \Lambda_{m,N} &:= \{K = (k_1, \dots, k_m) \in [0, N]^m \cap \mathbb{Z}^m : |K| = k_1 + \dots + k_m \leq N\}.\end{aligned}$$

It is not difficult to show that $|\Gamma_{m,N}| = |\Lambda_{m,N}| = \binom{N+m}{m}$.

The tautological line bundle $\mathcal{O}(-1)$ over the complex projective space \mathbb{CP}^m is a holomorphic line bundle with fibers

$$\mathcal{O}(-1)_{[x]} = \mathbb{C} \cdot x, \quad \forall [x] = [x_0 : \dots : x_m] \in \mathbb{CP}^m.$$

Its dual bundle, denoted by $\mathcal{O}(1)$, is called the hyperplane section bundle since $\mathcal{O}(1) = [H]$ where the divisor

$$H = \{[x] \in \mathbb{CP}^m : x_0 = 0\}$$

is a hyperplane in \mathbb{CP}^m . See [2] for details. By Theorem 15.5 in Chapter V of [1], $H^0(\mathbb{CP}^m, \mathcal{O}(N))$, the space of holomorphic sections of the tensor bundle $\mathcal{O}(N) = \mathcal{O}(1)^{\otimes N}$, is isomorphic to ${}^h\mathcal{P}_{m+1}^N$, the space of $(m+1)$ -variable homogenous polynomials of degree N . The Fubini-Study metric h_{FS} on $\mathcal{O}(1)$ can be described in the following way: over the open subset

$$U_0 = \{[x] = [x_0 : \dots : x_m] \in \mathbb{CP}^m : x_0 \neq 0\} \subset \mathbb{CP}^m,$$

we have a local frame of $\mathcal{O}(1)$

$$e([x]) = x_0.$$

Set

$$\|e([x])\|_{h_{\text{FS}}}^2 = \frac{|x_0|^2}{\sum_{i=0}^m |x_i|^2} = \frac{|x_0|^2}{\|x\|^2},$$

which is independent of the choice of representative x of $[x]$. In terms of affine coordinate

$$z = (z_1, \dots, z_m) = \left(\frac{x_1}{x_0}, \dots, \frac{x_m}{x_0} \right)$$

over U_0 ,

$$\|e(z)\|_{h_{\text{FS}}}^2 = (1 + \|z\|^2)^{-1} = \left(1 + \sum_{i=1}^m |z_i|^2\right)^{-1},$$

which defines a metric with positive Chern curvature form

$$\omega_{\text{FS}} = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \|e(z)\|_{h_{\text{FS}}}^2 = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log (1 + |z_1|^2 + \dots + |z_m|^2).$$

This induces a metric h_{FS}^N on the line bundle $\mathcal{O}(N)$ so that

$$\|e^{\otimes N}(z)\|_{h_{\text{FS}}^N}^2 = (1 + \|z\|^2)^{-N}.$$

With the frame $e^{\otimes N}$ over U_0 , for any $s \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$ which is represented by $p(x_0, \dots, x_m) \in {}^h\mathcal{P}_{m+1}^N$, we have

$$p(x_0, \dots, x_m) = \frac{p(x_0, \dots, x_m)}{x_0^N} e^{\otimes N}([x]) = p(1, z_1, \dots, z_m) e^{\otimes N}([x]),$$

which implies that all the elements in $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ can be viewed over U_0 as polynomials in (z_1, \dots, z_m) of degree at most N .

Since ω_{FS} is positive over \mathbb{CP}^m , we may take it as a polarized metric form on \mathbb{CP}^m and the associated volume form is $dV = \frac{\omega_{FS}^m}{m!}$. Thus, the metric h_{FS}^N together with the volume form dV induce a Hermitian inner product on the space of holomorphic sections $H^0(\mathbb{CP}^m, \mathcal{O}(N))$: $\forall s_1, s_2 \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$,

$$\langle\langle s_1, s_2 \rangle\rangle := \int_{\mathbb{CP}^m} \langle s_1, s_2 \rangle_{h_{FS}^N} dV.$$

With this inner product, there is an orthonormal basis $\{S_K^N\}_{K=(k_1, \dots, k_m) \in \Lambda_{m,N}}$, given in local affine coordinates (z_1, \dots, z_m) over U_0 by

$$S_K^N(z) = \sqrt{(N+1) \cdots (N+m)} \sqrt{\binom{N}{K}} z^K,$$

where we adopt the notations

$$\binom{N}{K} = \frac{N!}{(N-|K|)!k_1! \cdots k_m!}, \quad z^K := z_1^{k_1} \cdots z_m^{k_m}.$$

Thus $H^0(\mathbb{CP}^m, \mathcal{O}(N)) = \{s_N = \sum_{K \in \Lambda_{m,N}} c_K S_K^N : c = (c_K)_{K \in \Lambda_{m,N}} \in \mathbb{C}^{\binom{N+m}{m}}\}$. Endow $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ with the Gaussian probability measure γ_N defined by

$$d\gamma_N(s_N) := \pi^{-\binom{N+m}{m}} e^{-\|c\|^2} d_{2\binom{N+m}{m}} c,$$

where $\|c\|^2 = \sum_{K \in \Lambda_{m,N}} |c_K|^2$ and $d_{2\binom{N+m}{m}} c$ denotes the $2\binom{N+m}{m}$ -dimensional Lebesgue measure. γ_N is characterized by the property that $\{c_K\}_{K \in \Lambda_{m,N}}$ are independent and identically distributed(i.i.d.) standard complex Gaussian random variables. Then $(H^0(\mathbb{CP}^m, \mathcal{O}(N)), \gamma_N)$ is called the ensemble of $SU(m+1)$ Gaussian random polynomials of degree N as the random element s_N is distributional invariant under $SU(m+1)$ transformations of \mathbb{CP}^m . Its hole probability over the polydisc $(D(0, r))^m \subset \mathbb{C}^m$ is

$$\begin{aligned} P_{0,m}(r, N) &= \gamma_N \{s_N \in H^0(\mathbb{CP}^m, \mathcal{O}(N)) : 0 \notin s_N((\bar{D}(0, r))^m)\} \\ &= \pi^{-\binom{N+m}{m}} \int_{c \in \mathbb{C}^{\binom{N+m}{m}} : 0 \notin s_N((\bar{D}(0, r))^m)} e^{-\|c\|^2} d_{2\binom{N+m}{m}} c \\ &= \pi^{-\binom{N+m}{m}} \int_{c \in \mathbb{C}^{\binom{N+m}{m}} : 0 \notin \tilde{s}_N((\bar{D}(0, r))^m)} e^{-\|c\|^2} d_{2\binom{N+m}{m}} c, \end{aligned}$$

where $\tilde{s}_N(z) = \sum_{K \in \Lambda_{m,N}} c_K \sqrt{\binom{N}{K}} z^K$. Thereafter, when considering hole probability, we work on \tilde{s}_N instead of s_N for simplicity.

2. PRELIMINARIES

Definition 2.1. $Q_{r,m}(N) := \sum_{K \in \Lambda_{m,N}} \log \left[\binom{N}{K} r^{2|K|} \right]$

Lemma 2.2.

$$Q_{r,m}(N) = N^{m+1} \int_{\Sigma_m} E_r(x) d_m x + o(N^{m+1}) = \left[\frac{2m \log r}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k} \right] N^{m+1} + o(N^{m+1}).$$

Proof. We can prove inductively that for $k \geq 1$,

$$\left(\frac{k}{e}\right)^k \leq k! \leq \frac{k^{k+1}}{e^{k-1}} \Leftrightarrow k \log k - k \leq \log k! \leq (k+1) \log k - (k-1). \quad (2.1)$$

$$\Rightarrow -(k+1) \log N + (k-1) \leq k \log \frac{k}{N} - \log k! \leq -k \log N + k \text{ for } 0 \leq k \leq N. \quad (2.2)$$

$$\forall K = (k_1, \dots, k_m) \in \Lambda_{m,N},$$

$$\log \left[\binom{N}{K} r^{2|K|} \right] - N E_r \left(\frac{K}{N} \right) = \log N! + \sum_{i=1}^m (k_i \log \frac{k_i}{N} - \log k_i!) + [(N - |K|) \log \frac{N - |K|}{N} - \log (N - |K|)!],$$

Applying (2.1) and (2.2), we get

$$\log \left[\binom{N}{K} r^{2|K|} \right] - N E_r \left(\frac{K}{N} \right) \geq (N \log N - N) - (N + m + 1) \log N + (N - m - 1) = -(m + 1)(\log N + 1),$$

$$\log \left[\binom{N}{K} r^{2|K|} \right] - N E_r \left(\frac{K}{N} \right) \leq [(N + 1) \log N - (N - 1)] - N \log N + N = \log N + 1,$$

$$\Rightarrow \left| \log \left[\binom{N}{K} r^{2|K|} \right] - N E_r \left(\frac{K}{N} \right) \right| \leq (m + 1)(\log N + 1), \quad \forall K \in \Lambda_{m,N},$$

$$\begin{aligned} \Rightarrow \left| Q_{r,m}(N) - N \sum_{K \in \Lambda_{m,N}} E_r \left(\frac{K}{N} \right) \right| &\leq \sum_{K \in \Lambda_{m,N}} \left| \log \left[\binom{N}{K} r^{2|K|} \right] - N E_r \left(\frac{K}{N} \right) \right| \\ &\leq (m + 1)(\log N + 1) \binom{N + m}{m} = o(N^{m+1}). \end{aligned} \quad (2.3)$$

Take

$$\mathring{\Lambda}_{m,N} := \{K \in \Lambda_{m,N} : k_i \geq 1 \text{ for } 1 \leq i \leq m \text{ and } |K| \leq N - m - 1\} \subset \Lambda_{m,N}$$

and

$$\mathring{\Sigma}_m(N) := \bigcup_{K \in \mathring{\Lambda}_{m,N}} \left[\frac{k_1}{N}, \frac{k_1 + 1}{N} \right] \times \dots \times \left[\frac{k_m}{N}, \frac{k_m + 1}{N} \right] \subset \Sigma_m.$$

Then

$$\begin{aligned} |\mathring{\Lambda}_{m,N}| &= \binom{N - m - 1}{m}, \\ |\Lambda_{m,N} \setminus \mathring{\Lambda}_{m,N}| &= \binom{N + m}{m} - \binom{N - m - 1}{m} = O(N^{m-1}), \\ \text{Vol}_{\mathbb{R}^m}(\Sigma_m \setminus \mathring{\Sigma}_m(N)) &= \frac{1}{m!} - N^{-m} \binom{N - m - 1}{m} = O(N^{-1}). \end{aligned}$$

Over Σ_m we have

$$|E_r| \leq 2|\log r| + \frac{m+1}{e} = O(1),$$

so

$$\left| N \sum_{K \in \Lambda_{m,N}} E_r \left(\frac{K}{N} \right) - N \sum_{K \in \mathring{\Lambda}_{m,N}} E_r \left(\frac{K}{N} \right) \right| \leq N |\Lambda_{m,N} \setminus \mathring{\Lambda}_{m,N}| \sup_{\Sigma_m} |E_r| = O(N^m). \quad (2.4)$$

As

$$\sup_{\mathring{\Sigma}_m(N)} \|\nabla E_r\| \leq O(\log N),$$

$$\begin{aligned} &\left| N \sum_{K \in \mathring{\Lambda}_{m,N}} E_r \left(\frac{K}{N} \right) - N^{m+1} \int_{\mathring{\Sigma}_m(N)} E_r(x) d_m x \right| \\ &\leq N^{m+1} \sum_{K \in \mathring{\Lambda}_{m,N}} \int_{\left[\frac{k_1}{N}, \frac{k_1 + 1}{N} \right] \times \dots \times \left[\frac{k_m}{N}, \frac{k_m + 1}{N} \right]} \left| E_r \left(\frac{K}{N} \right) - E_r(x) \right| d_m x \\ &\leq N^{m+1} \binom{N - m - 1}{m} N^{-m} O(\log N) O(N^{-1}) \\ &= O(N^m \log N). \end{aligned} \quad (2.5)$$

$$\left| N^{m+1} \int_{\mathring{\Sigma}_m(N)} E_r(x) d_m x - N^{m+1} \int_{\Sigma_m} E_r(x) d_m x \right| \leq N^{m+1} \sup_{\Sigma_m} |E_r| \text{Vol}_{\mathbb{R}^m}(\Sigma_m \setminus \mathring{\Sigma}_m(N)) = O(N^m). \quad (2.6)$$

Combining (2.3)~(2.6), we thus obtain

$$\begin{aligned}
Q_{r,m}(N) &= N^{m+1} \int_{\Sigma_m} E_r(x) d_mx + o(N^{m+1}) \\
&= N^{m+1} \int_{\Sigma_m} 2 \sum_{i=1}^m x_i \log r - \left[\sum_{i=1}^m x_i \log x_i + \left(1 - \sum_{i=1}^m x_i\right) \log \left(1 - \sum_{i=1}^m x_i\right) \right] d_mx + o(N^{m+1}) \\
&= N^{m+1} \left[2m \log r \int_{\Sigma_m} x_1 d_mx - (m+1) \int_{\Sigma_m} x_1 \log x_1 d_mx \right] + o(N^{m+1}) \\
&= \left[\frac{2m \log r}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k} \right] N^{m+1} + o(N^{m+1}).
\end{aligned}$$

□

Remark 2.3. The scaled lattice $\frac{1}{N} \Lambda_{m,N} \subset \mathbb{R}^m$ will tend to Σ_m . Hence Lemma 2.2 is in fact converting a Riemann sum into a Riemann integral and estimating the error. Such procedures will appear several times in this paper.

Remark 2.4. The function $E_r(x)$ in the above lemma can also be written as $E_r(x) = -b_{\{x\}}(z_r) + \log(1 + \|z_r\|^2)$, where $z_r = (r, \dots, r) \in \mathbb{R}^m$ and $b_{\{x\}}$ is the exponential decay rate of the expected mass density of random L^2 normalized polynomials with some prescribed Newton polytope (see Theorem 1.2 and (78) in [8]).

Let $\xi = (\xi_1, \dots, \xi_m)$, where for $1 \leq i \leq m$, $\xi_i = (\xi_{i,0}, \dots, \xi_{i,N})$.

Definition 2.5. $W_{m,N}(\xi)$ is the $\binom{N+m}{m} \times \binom{N+m}{m}$ matrix with rows indexed by $\Gamma_{m,N}$ and columns indexed by $\Lambda_{m,N}$, such that $\forall J = (j_1, \dots, j_m) \in \Gamma_{m,N}$, $K = (k_1, \dots, k_m) \in \Lambda_{m,N}$, the (J, K) -entry of $W_{m,N}(\xi)$ is $\xi_J^K = \xi_{1,j_1}^{k_1} \dots \xi_{m,j_m}^{k_m}$.

Next lemma gives the formula for a ‘‘Vandermonde type’’ determinant.

Lemma 2.6. $|\det W_{m,N}(\xi)| = \prod_{i=1}^m \prod_{0 \leq j < k \leq N} |\xi_{i,j} - \xi_{i,k}|^{\binom{j+i-1}{i-1} \binom{N-k+m-i}{m-i}}.$

Proof. $\forall 1 \leq i \leq m$ and $0 \leq j < k \leq N$, the rows of $W_{m,N}(\xi)$ involving $\xi_{i,j}$ correspond to the set

$$\Gamma_{m,N}^{i,j} = \{(j_1, \dots, j_m) \in \Gamma_{m,N} : j_i = j\}$$

while those rows involving $\xi_{i,k}$ correspond to the set

$$\Gamma_{m,N}^{i,k} = \{(j_1, \dots, j_m) \in \Gamma_{m,N} : j_i = k\}. \quad (2.7)$$

Let

$$\tilde{\Gamma}_{m,N}^{i,j} = \{(j_1, \dots, \hat{j}_i, \dots, j_m) \in [0, N]^{m-1} \cap \mathbb{Z}^{m-1} : 0 \leq j_1 \leq \dots \leq j_{i-1} \leq j \leq j_{i+1} \leq \dots \leq j_m \leq N\},$$

$$\tilde{\Gamma}_{m,N}^{i,k} = \{(j_1, \dots, \hat{j}_i, \dots, j_m) \in [0, N]^{m-1} \cap \mathbb{Z}^{m-1} : 0 \leq j_1 \leq \dots \leq j_{i-1} \leq k \leq j_{i+1} \leq \dots \leq j_m \leq N\},$$

then

$$\begin{aligned}
|\Gamma_{m,N}^{i,j}| &= |\tilde{\Gamma}_{m,N}^{i,j}| = \binom{j+i-1}{i-1} \binom{N-j+m-i}{m-i}, \\
|\Gamma_{m,N}^{i,k}| &= |\tilde{\Gamma}_{m,N}^{i,k}| = \binom{k+i-1}{i-1} \binom{N-k+m-i}{m-i}.
\end{aligned}$$

Since $\forall 1 \leq i \leq m$,

$$\Gamma_{m,N} = \bigsqcup_{k=0}^N \Gamma_{m,N}^{i,k},$$

we thus have the equality

$$\sum_{k=0}^N \binom{k+i-1}{i-1} \binom{N-k+m-i}{m-i} = \binom{N+m}{m}. \quad (2.8)$$

$$\tilde{\Gamma}_{m,N}^{i,j} \cap \tilde{\Gamma}_{m,N}^{i,k} = \{(j_1, \dots, \hat{j}_i, \dots, j_m) \in [0, N]^{m-1} \cap \mathbb{Z}^{m-1} : 0 \leq j_1 \leq \dots \leq j_{i-1} \leq j < k \leq j_{i+1} \leq \dots \leq j_m \leq N\}$$

and

$$|\tilde{\Gamma}_{m,N}^{i,j} \cap \tilde{\Gamma}_{m,N}^{i,k}| = \binom{j+i-1}{i-1} \binom{N-k+m-i}{m-i},$$

which means that there are $\binom{j+i-1}{i-1} \binom{N-k+m-i}{m-i}$ pairs of rows, within each pair the only difference between two rows is replacing $\xi_{i,j}$ by $\xi_{i,k}$. Therefore, $\forall 1 \leq i \leq m$ and $\forall 0 \leq j < k \leq N$,

$$\begin{aligned} & (\xi_{i,j} - \xi_{i,k})^{\binom{j+i-1}{i-1} \binom{N-k+m-i}{m-i}} |\det W_{m,N}(\xi)|, \\ \Rightarrow G_{m,N}(\xi) &:= \prod_{i=1}^m \prod_{0 \leq j < k \leq N} (\xi_{i,j} - \xi_{i,k})^{\binom{j+i-1}{i-1} \binom{N-k+m-i}{m-i}} |\det W_{m,N}(\xi)|. \end{aligned} \quad (2.9)$$

$\forall 1 \leq i \leq m$,

$$\begin{aligned} \deg_{\xi_i} G_{m,N}(\xi) &= \sum_{0 \leq j < k \leq N} \binom{j+i-1}{i-1} \binom{N-k+m-i}{m-i} \\ &= \sum_{k=1}^N \left[\sum_{j=0}^{k-1} \binom{j+i-1}{i-1} \right] \binom{N-k+m-i}{m-i} \\ &= \sum_{k=1}^N \binom{k-1+i}{i} \binom{N-k+m-i}{m-i} \\ &= \sum_{k=1}^{N-1} \binom{(k-1)+(i+1)-1}{(i+1)-1} \binom{(N-1)-(k-1)+(m+1)-(i+1)}{(m+1)-(i+1)} \\ &= \binom{(N-1)+(m+1)}{m+1} \\ &= \binom{N+m}{m+1}, \end{aligned} \quad (2.10)$$

where the second to the last equality is due to (2.8). On the other hand, $\forall 1 \leq i \leq m$ and $1 \leq k \leq N$, the number of K's in $\Lambda_{m,N}$ with $k_i = k$ is $\binom{N-k+m-1}{m-1}$, so

$$\begin{aligned} \deg_{\xi_i} \det W_{m,N}(\xi) &= \sum_{k=1}^N k \binom{N-k+m-1}{m-1} \\ &= \binom{N+m}{m+1}, \end{aligned}$$

where the second equality is the special case $i = 1$ in (2.10).

$$\Rightarrow \deg_{\xi_i} \det W_{m,N}(\xi) = \deg_{\xi_i} G_{m,N}(\xi), \quad \forall 1 \leq i \leq m. \quad (2.11)$$

$$(2.9) \text{ and } (2.11) \Rightarrow \det W_{m,N}(\xi) = C_{m,N} G_{m,N} = C_{m,N} \prod_{i=1}^m \prod_{0 \leq j < k \leq N} (\xi_{i,j} - \xi_{i,k})^{\binom{j+i-1}{i-1} \binom{N-k+m-i}{m-i}},$$

where $C_{m,N}$ is a constant depending only on m and N . Consider the monomial

$$g_{m,N}(\xi) := \prod_{i=1}^m \prod_{k=1}^N \xi_{i,k}^{\sum_{j=0}^{k-1} \binom{j+i-1}{i-1} \binom{N-k+m-i}{m-i}} = \prod_{i=1}^m \prod_{k=1}^N \xi_{i,k}^{\binom{k+i-1}{i} \binom{N-k+m-i}{m-i}},$$

then

$$G_{m,N}(\xi) = \pm g_{m,N}(\xi) + \dots$$

In the appendix, we show that the coefficient of $g_{m,N}$ in the expansion of $\det W_{m,N}(\xi)$ equals 1, and therefore $C_{m,N} = \pm 1$. \square

3. LOWER BOUND IN THEOREM 0.1

Proof of the lower bound in Theorem 0.1.

$$|\tilde{s}_N(z)| \geq |c_{(0,\dots,0)}| - \sum_{K \in \Lambda_{m,N} \setminus \{(0,\dots,0)\}} |c_K| \sqrt{\binom{N}{K}} r^{|K|}, \quad \forall z = (z_1, \dots, z_m) \in (\bar{D}(0, r))^m. \quad (3.1)$$

Consider the event $\Omega_{r,m,N}$:

$$\begin{aligned} (i) \quad & |c_{(0,\dots,0)}| \geq \sqrt{N}, \\ (ii) \quad & |c_K| \leq \frac{1}{2\sqrt{N} \sqrt{\binom{N}{K}} r^{|K|} \binom{|K|+m-1}{m-1}}, \quad K \in \Lambda_{m,N} \setminus \{(0,\dots,0)\}. \end{aligned}$$

Then if $\Omega_{r,m,N}$ occurs, by (3.1), we have $\forall z = (z_1, \dots, z_m) \in (\bar{D}(0, r))^m$,

$$\begin{aligned} |\tilde{s}_N(z)| &\geq \sqrt{N} - \sum_{K \in \Lambda_{m,N} \setminus \{(0,\dots,0)\}} \frac{\sqrt{\binom{N}{K}} r^{|K|}}{2\sqrt{N} \sqrt{\binom{N}{K}} r^{|K|} \binom{|K|+m-1}{m-1}} \\ &= \sqrt{N} - \sum_{K \in \Lambda_{m,N} \setminus \{(0,\dots,0)\}} \frac{1}{2\sqrt{N} \binom{|K|+m-1}{m-1}} \\ &= \sqrt{N} - \sum_{k=1}^N \frac{1}{2\sqrt{N}} \\ &= \frac{1}{2}\sqrt{N} > 0, \end{aligned}$$

$$\Rightarrow P_{0,m}(r, N) \geq \gamma_N(\Omega_{r,m,N}) = \gamma_N(|c_{(0,\dots,0)}| \geq \sqrt{N}) \prod_{K \in \Lambda_{m,N} \setminus \{(0,\dots,0)\}} \gamma_N\left(|c_K| \leq \frac{1}{2\sqrt{N} \sqrt{\binom{N}{K}} r^{|K|} \binom{|K|+m-1}{m-1}}\right),$$

where $\gamma_N(|c_{(0,\dots,0)}| \geq \sqrt{N}) = e^{-N}$. Since $r \geq 1$, $\frac{1}{2\sqrt{N} \sqrt{\binom{N}{K}} r^{|K|} \binom{|K|+m-1}{m-1}} \leq 1$ for $K \in \Lambda_{m,N} \setminus \{(0,\dots,0)\}$,

$$\gamma_N\left(|c_K| \leq \frac{1}{2\sqrt{N} \sqrt{\binom{N}{K}} r^{|K|} \binom{|K|+m-1}{m-1}}\right) \geq \frac{1}{2} \left[\frac{1}{2\sqrt{N} \sqrt{\binom{N}{K}} r^{|K|} \binom{|K|+m-1}{m-1}} \right]^2 = \frac{1}{8N \binom{N}{K} r^{2|K|} \binom{|K|+m-1}{m-1}^2},$$

$$\log P_{0,m}(r, N) \geq -N - \sum_{K \in \Lambda_{m,N} \setminus \{(0,\dots,0)\}} \left\{ \log 8 + \log N + 2 \log \binom{|K|+m-1}{m-1} + \log \left[\binom{N}{K} r^{2|K|} \right] \right\},$$

where

$$\log \binom{|K|+m-1}{m-1} \leq \log \binom{N+m-1}{m-1} = O(\log N),$$

$$\Rightarrow \sum_{K \in \Lambda_{m,N} \setminus \{(0,\dots,0)\}} \left[\log 8 + \log N + 2 \log \binom{|K|+m-1}{m-1} \right] = \binom{N+m}{m} O(\log N) = o(N^{m+1}),$$

$$\begin{aligned} \Rightarrow \log P_{0,m}(r, N) &\geq - \sum_{K \in \Lambda_{m,N} \setminus \{(0,\dots,0)\}} \log \left[\binom{N}{K} r^{2|K|} \right] + o(N^{m+1}) \\ &= -Q_{r,m}(N) + o(N^{m+1}) = -N^{m+1} \int_{\Sigma_m} E_r(x) \, d_m x + o(N^{m+1}). \end{aligned}$$

□

4. UPPER BOUND IN THEOREM 0.1

Let $\delta > 0$ be small, $\kappa = 1 - \sqrt{\delta}$. We shall first treat δ as a small positive constant and at the end we will let $\delta \rightarrow 0+$. For the sake of clarity, all the constants C , capital O and little o terms listed throughout this paper will not depend on δ unless stated.

Definition 4.1. $z_j(N) := \kappa r e^{2\pi\sqrt{-1}\frac{j}{N+1}}$, for $0 \leq j \leq N$.

$\forall p \in \mathbb{Z}^+$, assume $N+1 = q(N)p + l(N)$, where $q(N) \in \mathbb{Z}$, $q(N) \geq 0$ and $0 \leq l(N) < p$. For convenience, we drop the dependence of N when there is no confusion. $\forall 1 \leq i \leq m$, assign the values of $\xi_i = (\xi_{i,0}, \dots, \xi_{i,N})$ by means of the table below:

$\xi_{i,0} = z_0$	\cdots	$\xi_{i,(q-1)p} = z_{q-1}$	$\xi_{i,qp} = z_q$
$\xi_{i,1} = z_{q+1}$	\cdots	$\xi_{i,(q-1)p+1} = z_{(q+1)+(q-1)}$	$\xi_{i,qp+1} = z_{(q+1)+q}$
$\dots\dots\dots$	\dots	$\dots\dots\dots$	$\dots\dots\dots$
$\xi_{i,l-1} = z_{(l-1)(q+1)}$	\cdots	$\xi_{i,(q-1)p+(l-1)} = z_{(l-1)(q+1)+(q-1)}$	$\xi_{i,qp+(l-1)} = z_{(l-1)(q+1)+q}$
$\xi_{i,l} = z_{l(q+1)}$	\cdots	$\xi_{i,(q-1)p+l} = z_{l(q+1)+(q-1)}$	
$\dots\dots\dots$	\dots	$\dots\dots\dots$	
$\xi_{i,p-1} = z_{l(q+1)+(p-1-l)q}$	\cdots	$\xi_{i,(q-1)p+(p-1)} = z_{l(q+1)+(p-1-l)q+(q-1)}$	

(4.1)

Intuitively, table (4.1) gives a way to choose points $\xi_{i,j}$ ($j = 0, 1, \dots$) one after another on the circle of radius κr that the arguments of each two consecutive points differ approximately by $\frac{2\pi}{p}$. Denote the bijection of $N+1$ letters $\{0, \dots, N\}$ indicated in table (4.1) by τ , i.e. $z_j = \xi_{i,\tau(j)}$ for $0 \leq j \leq N$ and $1 \leq i \leq m$. Denote

$$\begin{aligned}
I_0 &= \{0, \dots, q\}, \quad a_0 = 0, \\
I_1 &= \{q+1, \dots, (q+1)+q\}, \quad a_1 = q+1, \\
&\dots \\
I_{l-1} &= \{(l-1)(q+1), \dots, (l-1)(q+1)+q\}, \quad a_{l-1} = (l-1)(q+1), \\
I_l &= \{l(q+1), \dots, l(q+1)+(q-1)\}, \quad a_l = l(q+1), \\
&\dots \\
I_{p-1} &= \{l(q+1) + (p-1-l)q, \dots, l(q+1) + (p-1-l)q + (q-1)\}, \quad a_{p-1} = l(q+1) + (p-1-l)q.
\end{aligned}$$

I_0, \dots, I_{p-1} give a partition of $\{0, \dots, N\}$. Again there is an implicit dependence on N for each term defined above, and we would show this dependence explicitly when necessary. Then

$$\begin{aligned}
a_t &= tq + \min\{t, l\} = \begin{cases} t(q+1) & \text{when } j \in I_t, \quad 0 \leq t \leq l, \\ l(q+1) + (t-l)q & \text{when } j \in I_t, \quad l+1 \leq t \leq p-1, \end{cases} \\
\tau(j) &= (j - a_t)p + t = \begin{cases} [j - t(q+1)]p + t & \text{when } j \in I_t, \quad 0 \leq t \leq l, \\ [j - l(q+1) - (t-l)q]p + t & \text{when } j \in I_t, \quad l+1 \leq t \leq p-1, \end{cases}
\end{aligned}$$

and if $\{j(N)\}_{N=1}^\infty$ is a sequence satisfying $j(N) \in I_t(N)$, $\forall N \geq 1$,

$$\begin{aligned}
&|\tau_N(j(N)) - pj(N) + t(N+1)| \leq 2p^2, \\
&\Rightarrow \frac{\tau_N(j(N))}{N+1} - \left(p \frac{j(N)}{N+1} - t\right) = O(N^{-1}).
\end{aligned}
\tag{4.2}$$

Lemma 4.2. *With the assignment of the values of ξ_i given in (4.1),*

$$\log |\det W_{m,N}(\xi)| = m \binom{N+m}{m+1} \log(\kappa r) + \frac{\beta_m}{p} N^{m+1} + o(N^{m+1}),$$

where $\beta_m = \frac{1}{(m-1)!} \int_0^1 x^m \log[2 \sin(\pi x)] dx$, which is finite for each $m \geq 1$ by comparison test of improper integrals.

Proof. By Lemma 2.6,

$$\begin{aligned}
\log |\det W_{m,N}(\xi)| &= \log \left[\prod_{i=1}^m \prod_{0 \leq j < k \leq N} |\xi_{i,j} - \xi_{i,k}|^{\binom{j+i-1}{i-1} \binom{N-k+m-i}{m-i}} \right] \\
&= \sum_{i=1}^m \sum_{0 \leq j < k \leq N} \binom{j+i-1}{i-1} \binom{N-k+m-i}{m-i} \log \left| \frac{\xi_{i,j}}{\kappa r} - \frac{\xi_{i,k}}{\kappa r} \right| \\
&\quad + \sum_{i=1}^m \sum_{0 \leq j < k \leq N} \binom{j+i-1}{i-1} \binom{N-k+m-i}{m-i} \log(\kappa r) \\
&= \sum_{i=1}^m \sum_{0 \leq \tau(j) < \tau(k) \leq N} \binom{\tau(j)+i-1}{i-1} \binom{N-\tau(k)+m-i}{m-i} \log \left| \frac{\xi_{i,\tau(j)}}{\kappa r} - \frac{\xi_{i,\tau(k)}}{\kappa r} \right| \\
&\quad + m \binom{N+m}{m+1} \log(\kappa r) \\
&= \sum_{i=1}^m \sum_{0 \leq \tau(j) < \tau(k) \leq N} \binom{\tau(j)+i-1}{i-1} \binom{N-\tau(k)+m-i}{m-i} \log |e^{2\pi\sqrt{-1}\frac{j}{N+1}} - e^{2\pi\sqrt{-1}\frac{k}{N+1}}| \\
&\quad + m \binom{N+m}{m+1} \log(\kappa r)
\end{aligned}$$

where the second part of the third equality is due to (2.10). We are going to show that the sum in the last equality can be approximated by a double integral.

$$\begin{aligned}
&\sum_{i=1}^m \sum_{0 \leq \tau(j) < \tau(k) \leq N} \binom{\tau(j)+i-1}{i-1} \binom{N-\tau(k)+m-i}{m-i} \log |e^{2\pi\sqrt{-1}\frac{j}{N+1}} - e^{2\pi\sqrt{-1}\frac{k}{N+1}}| \\
&= \sum_{i=1}^m \sum_{0 \leq \tau(j) < \tau(k) \leq N} \left[\frac{(\tau(j))^{i-1}}{(i-1)!} + o((\tau(j))^{i-1}) \right] \left[\frac{(N-\tau(k))^{m-i}}{(m-i)!} + o((N-\tau(k))^{m-i}) \right] \log |1 - e^{2\pi\sqrt{-1}(\frac{j}{N+1} - \frac{k}{N+1})}|,
\end{aligned} \tag{4.3}$$

$\forall 1 \leq i \leq m, 0 \leq u, v \leq p-1$, denote

$$\begin{aligned}
L_{u,v,N} &= \{(j, k) \in I_u \times I_v : \tau(j) < \tau(k)\}, \\
T_{u,v}(N) &= \bigcup_{(j,k) \in L_{u,v,N}} \left[\frac{j}{N+1}, \frac{j+1}{N+1} \right] \times \left[\frac{k}{N+1}, \frac{k+1}{N+1} \right], \\
\mathring{L}_{u,v,N} &= \{(j, k) \in L_{u,v,N} : j - k \neq \pm N \text{ and } j - k \neq \pm 1\} \subset L_{u,v,N}, \\
\mathring{T}_{u,v}(N) &= \bigcup_{(j,k) \in \mathring{L}_{u,v,N}} \left[\frac{j}{N+1}, \frac{j+1}{N+1} \right] \times \left[\frac{k}{N+1}, \frac{k+1}{N+1} \right] \subset T_{u,v}(N),
\end{aligned}$$

and a function defined over $\{(x, y) \in (0, 1) \times (0, 1) : x \neq y\}$:

$$g_{u,v}^i(x, y) = (px - u)^{i-1} [1 - (py - v)]^{m-i} \log |1 - e^{2\pi\sqrt{-1}(x-y)}|.$$

Then

$$|L_{u,v,N} \setminus \mathring{L}_{u,v,N}| \leq 2N + 2, \tag{4.4}$$

$$\text{Vol}_{\mathbb{R}^2}(T_{u,v}(N) \setminus \mathring{T}_{u,v}(N)) \leq O(N^{-1}), \tag{4.5}$$

$$\frac{1}{N+1} \leq \left| \frac{j-k}{N+1} \right| \leq \frac{N}{N+1} \text{ for } (j, k) \in L_{u,v,N}, \tag{4.6}$$

$$\frac{1}{N+1} \leq |x-y| \leq \frac{N}{N+1} \text{ for } (x, y) \in \mathring{T}_{u,v}(N), \tag{4.7}$$

$$|g_{u,v}^i(x, y)| \leq O(\log N) \text{ if } \frac{1}{N+1} \leq |x-y| \leq \frac{N}{N+1}, \tag{4.8}$$

$$\|\nabla g_{u,v}^i(x, y)\| \leq O(N^{\frac{1}{2}}) \text{ if } \frac{1}{\sqrt{N+1}} \leq |x-y| \leq 1 - \frac{1}{\sqrt{N+1}}. \quad (4.9)$$

From (4.2), we have

$$\begin{aligned} & \sum_{0 \leq \tau(j) < \tau(k) \leq N} (\tau(j))^{i-1} (N - \tau(k))^{m-i} \log |1 - e^{2\pi\sqrt{-1}(\frac{j}{N+1} - \frac{k}{N+1})}| \\ &= (N+1)^{m-1} \\ & \times \sum_{0 \leq u, v \leq p-1} \sum_{(j,k) \in \dot{L}_{u,v,N}} [p\frac{j}{N+1} - u + O(N^{-1})]^{i-1} [1 - (p\frac{k}{N+1} - v) + O(N^{-1})]^{m-i} \log |1 - e^{2\pi\sqrt{-1}(\frac{j}{N+1} - \frac{k}{N+1})}|, \end{aligned} \quad (4.10)$$

$\forall 0 \leq u, v \leq p-1$, by (4.4), (4.6) and (4.8),

$$\begin{aligned} & \sum_{(j,k) \in \dot{L}_{u,v,N}} (p\frac{j}{N+1} - u)^{i-1} [1 - (p\frac{k}{N+1} - v)]^{m-i} \log |1 - e^{2\pi\sqrt{-1}(\frac{j}{N+1} - \frac{k}{N+1})}| \\ &= \sum_{(j,k) \in \dot{L}_{u,v,N}} g_{u,v}^i(\frac{j}{N+1}, \frac{k}{N+1}) \\ &= \sum_{(j,k) \in \dot{L}_{u,v,N}} g_{u,v}^i(\frac{j}{N+1}, \frac{k}{N+1}) + O(N \log N), \end{aligned} \quad (4.11)$$

$$\begin{aligned} & |(N+1)^{-2} \sum_{(j,k) \in \dot{L}_{u,v,N}} g_{u,v}^i(\frac{j}{N+1}, \frac{k}{N+1}) - \iint_{\dot{T}_{u,v}(N)} g_{u,v}^i(x, y) \, dx dy| \\ &\leq \sum_{(j,k) \in \dot{L}_{u,v,N}} \iint_{[\frac{j}{N+1}, \frac{j+1}{N+1}] \times [\frac{k}{N+1}, \frac{k+1}{N+1}]} |g_{u,v}^i(x, y) - g_{u,v}^i(\frac{j}{N+1}, \frac{k}{N+1})| \, dx dy \\ &= \sum_{(j,k) \in \dot{L}_{u,v,N} : \frac{1}{\sqrt{N+1}} \leq |\frac{j-k}{N+1}| \leq 1 - \frac{1}{\sqrt{N+1}}} \iint_{[\frac{j}{N+1}, \frac{j+1}{N+1}] \times [\frac{k}{N+1}, \frac{k+1}{N+1}]} |g_{u,v}^i(x, y) - g_{u,v}^i(\frac{j}{N+1}, \frac{k}{N+1})| \, dx dy \\ &+ \sum_{(j,k) \in \dot{L}_{u,v,N} : |\frac{j-k}{N+1}| < \frac{1}{\sqrt{N+1}} \text{ or } |\frac{j-k}{N+1}| > 1 - \frac{1}{\sqrt{N+1}}} \iint_{[\frac{j}{N+1}, \frac{j+1}{N+1}] \times [\frac{k}{N+1}, \frac{k+1}{N+1}]} |g_{u,v}^i(x, y) - g_{u,v}^i(\frac{j}{N+1}, \frac{k}{N+1})| \, dx dy. \end{aligned} \quad (4.12)$$

Since

$$|(j, k) \in \dot{L}_{u,v,N} : \frac{1}{\sqrt{N+1}} \leq |\frac{j-k}{N+1}| \leq 1 - \frac{1}{\sqrt{N+1}}| \leq |\dot{L}_{u,v,N}| = O(N^2),$$

$$|(j, k) \in \dot{L}_{u,v,N} : |\frac{j-k}{N+1}| < \frac{1}{\sqrt{N+1}} \text{ or } |\frac{j-k}{N+1}| > 1 - \frac{1}{\sqrt{N+1}}| \leq O(N^{\frac{3}{2}}),$$

$$\begin{aligned} (4.9) \Rightarrow & \sum_{(j,k) \in \dot{L}_{u,v,N} : \frac{1}{\sqrt{N+1}} \leq |\frac{j-k}{N+1}| \leq 1 - \frac{1}{\sqrt{N+1}}} \iint_{[\frac{j}{N+1}, \frac{j+1}{N+1}] \times [\frac{k}{N+1}, \frac{k+1}{N+1}]} |g_{u,v}^i(x, y) - g_{u,v}^i(\frac{j}{N+1}, \frac{k}{N+1})| \, dx dy \\ & \leq O(N^2) \times (N+1)^{-2} \times \frac{\sqrt{2}}{N+1} \times \sup_{\frac{1}{\sqrt{N+1}} \leq |x-y| \leq 1 - \frac{1}{\sqrt{N+1}}} \|\nabla g_{u,v}^i(x, y)\| \\ & \leq O(N^{-\frac{1}{2}}), \end{aligned} \quad (4.13)$$

and by (4.7), (4.8),

$$\begin{aligned}
& \sum_{(j,k) \in \tilde{L}_{u,v,N} : |\frac{j-k}{N+1}| < \frac{1}{\sqrt{N+1}} \text{ or } |\frac{j-k}{N+1}| > 1 - \frac{1}{\sqrt{N+1}}} \iint_{[\frac{j}{N+1}, \frac{j+1}{N+1}] \times [\frac{k}{N+1}, \frac{k+1}{N+1}]} |g_{u,v}^i(x, y) - g_{u,v}^i(\frac{j}{N+1}, \frac{k}{N+1})| dx dy \\
& \leq O(N^{\frac{3}{2}}) \times (N+1)^{-2} \times O(\log N) \\
& = O(N^{-\frac{1}{2}} \log N).
\end{aligned} \tag{4.14}$$

Denote $T_{u,v} = \{(x, y) \in \mathbb{R}^2 : 0 \leq x - \frac{u}{p} \leq y - \frac{v}{p} \leq \frac{1}{p}\}$. Since $g_{u,v}^i$ is L_{loc}^1 , the measure $g_{u,v}^i(x, y) dx dy$ is absolutely continuous with respect to the Lebesgue measure. Thus by lemma 4.3 below, we have

$$\iint_{T_{u,v}(N)} g_{u,v}^i(x, y) dx dy - \iint_{T_{u,v}} g_{u,v}^i(x, y) dx dy = o(1) \text{ as } N \rightarrow \infty. \tag{4.15}$$

$$\begin{aligned}
(4.11) \sim (4.15) & \Rightarrow \sum_{(j,k) \in L_{u,v,N}} (p \frac{j}{N+1} - u)^{i-1} [1 - (p \frac{k}{N+1} - v)]^{m-i} \log |1 - e^{2\pi\sqrt{-1}(\frac{j}{N+1} - \frac{k}{N+1})}| \\
& = (N+1)^2 \iint_{T_{u,v}} g_{u,v}^i(x, y) dx dy + o(N^2).
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
(4.16) + (4.10) & \Rightarrow \sum_{0 \leq \tau(j) < \tau(k) \leq N} (\tau(j))^{i-1} (N - \tau(k))^{m-i} \log |1 - e^{2\pi\sqrt{-1}(\frac{j}{N+1} - \frac{k}{N+1})}| \\
& = (N+1)^{m+1} \sum_{0 \leq u, v \leq p-1} \iint_{T_{u,v}} g_{u,v}^i(x, y) dx dy + o(N^{m+1}),
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
(4.17) + (4.3) & \Rightarrow \sum_{i=1}^m \sum_{0 \leq \tau(j) < \tau(k) \leq N} \binom{\tau(j) + i - 1}{i-1} \binom{N - \tau(k) + m - i}{m-i} \log |e^{2\pi\sqrt{-1}\frac{j}{N+1}} - e^{2\pi\sqrt{-1}\frac{k}{N+1}}| \\
& = \sum_{i=1}^m \sum_{0 \leq u, v \leq p-1} \iint_{T_{u,v}} \frac{g_{u,v}^i(x, y)}{(i-1)!(m-i)!} dx dy + o(N^{m+1}) \\
& = \sum_{i=1}^m \sum_{0 \leq u, v \leq p-1} \iint_{T_{u,v}} \frac{[p(x - \frac{u}{p})]^{i-1}}{(i-1)!} \frac{[1 - p(y - \frac{v}{p})]^{m-i}}{(m-i)!} \log |1 - e^{2\pi\sqrt{-1}(x-y)}| dx dy + o(N^{m+1}) \\
& = \sum_{i=1}^m \sum_{0 \leq u, v \leq p-1} \iint_{T_{0,0}} \frac{(px)^{i-1}}{(i-1)!} \frac{(1-py)^{m-i}}{(m-i)!} \log |1 - e^{2\pi\sqrt{-1}(x-y+\frac{u}{p}-\frac{v}{p})}| dx dy + o(N^{m+1}) \\
& = \sum_{i=1}^m \sum_{0 \leq u \leq p-1} \iint_{T_{0,0}} \frac{(px)^{i-1}}{(i-1)!} \frac{(1-py)^{m-i}}{(m-i)!} \log \left[\prod_{v=0}^{p-1} |e^{2\pi\sqrt{-1}\frac{u}{p}} - e^{2\pi\sqrt{-1}(x-y+\frac{u}{p})}| \right] dx dy + o(N^{m+1}) \\
& = p \sum_{i=1}^m \iint_{T_{0,0}} \frac{(px)^{i-1}}{(i-1)!} \frac{(1-py)^{m-i}}{(m-i)!} \log |1 - e^{2\pi\sqrt{-1}(px-py)}| dx dy + o(N^{m+1}) \\
& = \frac{1}{p} \iint_T \sum_{i=1}^m \frac{x^{i-1}}{(i-1)!} \frac{(1-y)^{m-i}}{(m-i)!} \log |1 - e^{2\pi\sqrt{-1}(x-y)}| dx dy + o(N^{m+1}) \\
& = \frac{1}{p(m-1)!} \iint_T (1+x-y)^{m-1} \log |1 - e^{2\pi\sqrt{-1}(x-y)}| dx dy + o(N^{m+1}),
\end{aligned}$$

where $T = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq 1\}$. Make change of variables: $\tilde{x} = x - y$, $\tilde{y} = y$, then T is mapped to $\tilde{T} = \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 : -1 \leq \tilde{x} \leq 0, -\tilde{x} \leq \tilde{y} \leq 1\}$.

$$\begin{aligned}
& \frac{1}{(m-1)!} \iint_T (1+x-y)^{m-1} \log |1 - e^{2\pi\sqrt{-1}(x-y)}| \, dx dy \\
&= \frac{1}{(m-1)!} \iint_{\tilde{T}} (1+\tilde{x})^{m-1} \log |1 - e^{2\pi\sqrt{-1}\tilde{x}}| \, d\tilde{x} d\tilde{y} \\
&= \frac{1}{(m-1)!} \int_{-1}^0 (1+\tilde{x})^m \log |1 - e^{2\pi\sqrt{-1}\tilde{x}}| \, d\tilde{x} \\
&= \frac{1}{(m-1)!} \int_0^1 x^m \log |1 - e^{2\pi\sqrt{-1}x}| \, dx \\
&= \frac{1}{(m-1)!} \int_0^1 x^m \log [2 \sin(\pi x)] \, dx \\
&= \beta_m, \\
&\Rightarrow \sum_{i=1}^m \sum_{0 \leq \tau(j) < \tau(k) \leq N} \binom{\tau(j)+i-1}{i-1} \binom{N-\tau(k)+m-i}{m-i} \log |e^{2\pi\sqrt{-1}\frac{j}{N+1}} - e^{2\pi\sqrt{-1}\frac{k}{N+1}}| = \frac{\beta_m}{p} N^{m+1} + o(N^{m+1}), \\
&\Rightarrow \log |\det W_{m,N}(\xi)| = m \binom{N+m}{m+1} \log(\kappa r) + \frac{\beta_m}{p} N^{m+1} + o(N^{m+1}).
\end{aligned}$$

□

Lemma 4.3. $\lim_{N \rightarrow \infty} \text{Vol}_{\mathbb{R}^2}(T_{u,v} \triangle \tilde{T}_{u,v}(N)) = 0$ for any $0 \leq u, v \leq p-1$.

Proof. By (4.5), it is equivalent to show that $\lim_{N \rightarrow \infty} \text{Vol}_{\mathbb{R}^2}(T_{u,v} \triangle T_{u,v}(N)) = 0$, which is a direct consequence of $\lim_{N \rightarrow \infty} T_{u,v}(N) \setminus \partial T_{u,v} = \tilde{T}_{u,v}$.

First let's show $\limsup_{N \rightarrow \infty} T_{u,v}(N) \subset T_{u,v}$. $\forall (x, y) \in \limsup_{N \rightarrow \infty} T_{u,v}(N)$, $\exists \{N_n\}_{n=1}^\infty \rightarrow \infty$ such that $\forall n \geq 1$, $\exists (j(N_n), k(N_n)) \in I_u(N_n) \times I_v(N_n)$, $\tau_{N_n}(j(N_n)) < \tau_{N_n}(k(N_n))$ and $(x, y) \in [\frac{j(N_n)}{N_n+1}, \frac{j(N_n)+1}{N_n+1}] \times [\frac{k(N_n)}{N_n+1}, \frac{k(N_n)+1}{N_n+1}]$. Then $\lim_{n \rightarrow \infty} \frac{j(N_n)}{N_n+1} = x$, $\lim_{n \rightarrow \infty} \frac{k(N_n)}{N_n+1} = y$. Since $0 \leq \frac{\tau_{N_n}(j(N_n))}{N_n+1} < \frac{\tau_{N_n}(k(N_n))}{N_n+1} \leq \frac{N_n}{N_n+1}$ and $(j(N_n), k(N_n)) \in I_u(N_n) \times I_v(N_n)$, (4.2) implies that $0 \leq p \lim_{n \rightarrow \infty} \frac{j(N_n)}{N_n+1} - u \leq p \lim_{n \rightarrow \infty} \frac{k(N_n)}{N_n+1} - v \leq 1$. Hence $0 \leq px - u \leq py - v \leq 1$ and $(x, y) \in T_{u,v}$.

Next we will prove $\tilde{T}_{u,v} \subset \liminf_{N \rightarrow \infty} T_{u,v}(N)$. $\forall (x, y) \in \tilde{T}_{u,v}$, $0 < x - \frac{u}{p} < y - \frac{v}{p} < \frac{1}{p}$. Then there exists $0 < \epsilon_1, \epsilon_2, \eta_1, \eta_2 < \frac{1}{p}$ such that $x = \frac{u}{p} + \epsilon_1 = \frac{u+1}{p} - \eta_1$ and $y = \frac{v}{p} + \epsilon_2 = \frac{v+1}{p} - \eta_2$. For each $N > 0$, define $j(N) = \lfloor (N+1)x \rfloor$ and $k(N) = \lfloor (N+1)y \rfloor$. When N is large enough, $j(N) = \lfloor (N+1)(\frac{u}{p} + \epsilon_1) \rfloor = uq(N) + \lfloor u\frac{l(N)}{p} + \epsilon_1(N+1) \rfloor \geq uq(N) + \min\{u, l(N)\} = a_u$, while $j(N) = \lfloor (N+1)(\frac{u+1}{p} - \eta_1) \rfloor = (u+1)q(N) + \lfloor (u+1)\frac{l(N)}{p} - \eta_1(N+1) \rfloor \leq (u+1)q(N) + \min\{u+1, l(N)\} - 1 = a_{u+1} - 1$ for $0 \leq u < p-1$, which indicates that $j(N) \in I_u(N)$. And similarly, $k(N) \in I_v(N)$ for N large. Moreover, $\lim_{N \rightarrow \infty} \frac{\tau(j(N))}{N+1} = p \lim_{N \rightarrow \infty} \frac{j(N)}{N+1} - u = p \lim_{N \rightarrow \infty} \frac{\lfloor (N+1)x \rfloor}{N+1} - u = px - u$, similarly $\lim_{N \rightarrow \infty} \frac{\tau(k(N))}{N+1} = py - v$. And since $0 < px - u < py - v < 1$, for N large enough, $0 < \frac{\tau(j(N))}{N+1} < \frac{\tau(k(N))}{N+1} < 1 \Rightarrow 0 < \tau(j(N)) < \tau(k(N)) \leq N$. Thus by the definition of $j(N)$ and $k(N)$, we have, for N large, $(x, y) \in [\frac{j(N)}{N+1}, \frac{j(N)+1}{N+1}] \times [\frac{k(N)}{N+1}, \frac{k(N)+1}{N+1}] \subset \bigcup_{(j,k) \in L_{u,v,N}} [\frac{j}{N+1}, \frac{j+1}{N+1}] \times [\frac{k}{N+1}, \frac{k+1}{N+1}] = T_{u,v}(N)$, which implies that $(x, y) \in \liminf_{N \rightarrow \infty} T_{u,v}(N)$.

Inclusion, we have

$$\begin{aligned} \mathring{T}_{u,v} &\subset \liminf_{N \rightarrow \infty} T_{u,v}(N) \subset \limsup_{N \rightarrow \infty} T_{u,v}(N) \subset T_{u,v}, \\ &\Rightarrow \lim_{N \rightarrow \infty} T_{u,v}(N) \setminus \partial T_{u,v} = \mathring{T}_{u,v}. \end{aligned}$$

□

Let $\zeta = (\zeta_J)_{J \in \Gamma_{m,N}}^t = (\tilde{s}_N(\xi_J))_{J \in \Gamma_{m,N}}^t = (\tilde{s}_N(\xi_{1,j_1}, \dots, \xi_{m,j_m}))_{J \in \Gamma_{m,N}}^t$ be a dimension $\binom{N+m}{m}$ mean zero complex Gaussian random vector. Denote its covariance matrix by Σ , then $\forall J = (j_1, \dots, j_m), J' = (j'_1, \dots, j'_m) \in \Gamma_{m,N}$,

$$\begin{aligned} \Sigma_{J,J'} &= \mathbb{E}_N(\zeta_J \bar{\zeta}_{J'}) = \mathbb{E}_N(\tilde{s}_N(\xi_J) \overline{\tilde{s}_N(\xi_{J'})}) \\ &= \sum_{K \in \Lambda_{m,N}} [\sqrt{\binom{N}{K}} \xi_J^K] [\sqrt{\binom{N}{K}} \bar{\xi}_{J'}^K] \\ &= \sum_{K \in \Lambda_{m,N}} \binom{N}{K} (\xi_J \bar{\xi}_{J'})^K \\ &= (1 + \xi_J \bar{\xi}_{J'})^N \\ &= (1 + \xi_{1,j_1} \bar{\xi}_{1,j'_1} + \dots + \xi_{m,j_m} \bar{\xi}_{m,j'_m})^N, \end{aligned}$$

where \mathbb{E}_N denotes the expectation with respect to the probability measure γ_N .

Lemma 4.4. *With the assignment of ξ as in table (4.1),*

$$\log(\det \Sigma) = Q_{\kappa r, m}(N) + \frac{2\beta_m}{p} N^{m+1} + o(N^{m+1}).$$

Proof.

$$\Sigma = V_{m,N}(\xi) V_{m,N}^*(\xi),$$

where $V_{m,N}(\xi) = (\sqrt{\binom{N}{K}} \xi_J^K)_{J \in \Gamma_{m,N}, K \in \Lambda_{m,N}}$ is an $\binom{N+m}{m} \times \binom{N+m}{m}$ matrix.

$$\Rightarrow \det \Sigma = |\det V_{m,N}(\xi)|^2 = \prod_{K \in \Lambda_{m,N}} \binom{N}{K} |\det W_{m,N}(\xi)|^2$$

By Lemma 4.2,

$$\begin{aligned} \log(\det \Sigma) &= \sum_{K \in \Lambda_{m,N}} \log \binom{N}{K} + 2 \log |\det W_{m,N}(\xi)| \\ &= \sum_{K \in \Lambda_{m,N}} \log \binom{N}{K} + 2m \binom{N+m}{m+1} \log(\kappa r) + \frac{2\beta_m}{p} N^{m+1} + o(N^{m+1}) \\ &= \sum_{K \in \Lambda_{m,N}} \log \binom{N}{K} + 2 \sum_{K \in \Lambda_{m,N}} |K| \log(\kappa r) + \frac{2\beta_m}{p} N^{m+1} + o(N^{m+1}) \\ &= Q_{\kappa r, m}(N) + \frac{2\beta_m}{p} N^{m+1} + o(N^{m+1}). \end{aligned}$$

□

As $\log |\tilde{s}_N(z)|$ is plurisubharmonic in a neighbourhood of $(\bar{D}(0, r))^m$, we have

$$\begin{aligned}
& \log \prod_{J \in \Gamma_{m,N}} |\zeta_J| \\
&= \sum_{J \in \Gamma_{m,N}} \log |\tilde{s}_N(\xi_J)| \\
&\leq \sum_{J \in \Gamma_{m,N}} \int_{\partial D(0,r)} \cdots \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| \prod_{i=1}^m P_r(\xi_{i,j_i}, u_i) d\sigma_r(u_1) \cdots d\sigma_r(u_m) \\
&= (N+1)^m \int_{\partial D(0,r)} \cdots \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| \left[\sum_{J \in \Gamma_{m,N}} \prod_{i=1}^m \frac{P_r(\xi_{i,j_i}, u_i)}{N+1} - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d_mx \right] \\
&\quad d\sigma_r(u_1) \cdots d\sigma_r(u_m) \\
&\quad + (N+1)^m \int_{\partial D(0,r)} \cdots \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d_mx d\sigma_r(u_1) \cdots d\sigma_r(u_m) \\
&= I + II,
\end{aligned} \tag{4.18}$$

where $P_r(\xi, u) = \frac{r^2 - |\xi|^2}{|u - \xi|^2}$ is the Poisson kernel of $D(0, r)$, $d\sigma_r$ is the Haar measure on $\partial D(0, r)$, d_mx is the Lebesgue measure on \mathbb{R}^m , and

$$\begin{aligned}
H &= \bigcup_{0 \leq t_1, \dots, t_m \leq p-1} H_{t_1, \dots, t_m} := \bigcup_{0 \leq t_1, \dots, t_m \leq p-1} \{x = (x_1, \dots, x_m) \in \mathbb{R}^m : 0 \leq x_1 - \frac{t_1}{p} \leq \cdots \leq x_m - \frac{t_m}{p} \leq \frac{1}{p}\}. \\
I &\leq (N+1)^m \max_{u \in (\partial D(0,r))^m} \left| \sum_{J \in \Gamma_{m,N}} \prod_{i=1}^m \frac{P_r(\xi_{i,j_i}, u_i)}{N+1} - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d_mx \right| \\
&\quad \times \int_{\partial D(0,r)} \cdots \int_{\partial D(0,r)} |\log |\tilde{s}_N(u)|| d\sigma_r(u_1) \cdots d\sigma_r(u_m).
\end{aligned} \tag{4.19}$$

First let's estimate $\int_{\partial D(0,r)} \cdots \int_{\partial D(0,r)} |\log |\tilde{s}_N(u)|| d\sigma_r(u_1) \cdots d\sigma_r(u_m)$.

Lemma 4.5. $\gamma_N \left(\sup_{u \in (\partial D(0,r))^m} |\tilde{s}_N(u)| < 1 \right) \leq e^{-Q_{r,m}(N)}.$

Proof.

$$\begin{aligned}
\tilde{s}_N(u) &= \sum_{K \in \Lambda_{m,N}} c_K \sqrt{\binom{N}{K}} u^K \\
&\Rightarrow \frac{\partial^K}{\partial u^K} \tilde{s}_N(0) = K! \sqrt{\binom{N}{K}} c_K,
\end{aligned}$$

where $\frac{\partial^K}{\partial u^K}$ refers to $\frac{\partial^{k_1}}{\partial u_1^{k_1}} \cdots \frac{\partial^{k_m}}{\partial u_m^{k_m}}$ and $K! = k_1! \cdots k_m!$.

By Cauchy's integral formula,

$$\begin{aligned}
\frac{\partial^K}{\partial u^K} \tilde{s}_N(0) &= \frac{K!}{(2\pi\sqrt{-1})^m} \int_{\partial D(0,r)} \cdots \int_{\partial D(0,r)} \frac{\tilde{s}_N(u)}{\prod_{i=1}^m u_i^{k_i+1}} du_1 \cdots du_m, \\
&\Rightarrow c_K = \binom{N}{K}^{-\frac{1}{2}} \frac{1}{(2\pi\sqrt{-1})^m} \int_{\partial D(0,r)} \cdots \int_{\partial D(0,r)} \frac{\tilde{s}_N(u)}{\prod_{i=1}^m u_i^{k_i+1}} du_1 \cdots du_m, \\
&\Rightarrow |c_K| \leq \frac{\sup_{u \in (\partial D(0,r))^m} |\tilde{s}_N(u)|}{\sqrt{\binom{N}{K}} r^{|K|}}, \quad \forall K \in \Lambda_{m,N}.
\end{aligned}$$

Therefore, $\sup_{u \in (\partial D(0,r))^m} |\tilde{s}_N(u)| < 1$ would imply that $\forall K \in \Lambda_{m,N}$,

$$|c_K| \leq \left[\binom{N}{K} r^{2|K|} \right]^{-\frac{1}{2}}.$$

$$\begin{aligned} \Rightarrow \gamma_N \left(\sup_{u \in (\partial D(0,r))^m} |\tilde{s}_N(u)| < 1 \right) &\leq \prod_{K \in \Lambda_{m,N}} \gamma_N \left(|c_K| \leq \left[\binom{N}{K} r^{2|K|} \right]^{-\frac{1}{2}} \right) \\ &\leq \prod_{K \in \Lambda_{m,N}} \left[\binom{N}{K} r^{2|K|} \right]^{-1} \\ &= e^{-Q_{r,m}(N)}. \end{aligned}$$

□

The next lemma follows directly from the first part of Theorem 3.1 in [9]. But here we provide a self-contained proof without using the language of sections and metrics.

Lemma 4.6. *Given $U \subset \mathbb{C}^m$ open and bounded with $\sup_{z \in \bar{U}} |z| = R > 0$, then $\forall \eta > 0$,*

$$\gamma_N \{ \sup_{z \in \bar{U}} |\tilde{s}_N(z)| > (1 + R^2)^{\frac{N}{2}} e^{\eta N} \} \leq e^{-e^{\eta N}}, \text{ for } N \gg 1.$$

Proof. By Cauchy-Schwartz inequality,

$$\begin{aligned} \sup_{z \in \bar{U}} |\tilde{s}_N(z)| &= \sup_{z \in \bar{U}} \left| \sum_{K \in \Lambda_{m,N}} c_K \sqrt{\binom{N}{K}} z^K \right| \\ &\leq |c| \sup_{z \in \bar{U}} \left[\sum_{K \in \Lambda_{m,N}} \binom{N}{K} |z|^{2K} \right]^{\frac{1}{2}} \\ &= |c| \sup_{z \in \bar{U}} (1 + |z|^2)^{\frac{N}{2}} \\ &= |c| (1 + R^2)^{\frac{N}{2}}, \\ \Rightarrow \gamma_N \{ \sup_{z \in \bar{U}} |\tilde{s}_N(z)| > (1 + R^2)^{\frac{N}{2}} e^{\eta N} \} \\ &\leq \gamma_N \{ |c| > e^{\eta N} \} \\ &= e^{-e^{2\eta N} \sum_{k=0}^{(N+m)-1} \frac{e^{(2\eta N)k}}{k!}}, \\ \Rightarrow \log \gamma_N \{ \sup_{z \in \bar{U}} |\tilde{s}_N(z)| > (1 + R^2)^{\frac{N}{2}} e^{\eta N} \} \\ &\leq -e^{2\eta N} + \log \binom{N+m}{m} + (2\eta N) \left[\binom{N+m}{m} - 1 \right] \\ &\leq -e^{\eta N}, \text{ for } N \gg 1. \end{aligned}$$

□

Lemma 4.7. $\int_{\partial D(0,r)} \cdots \int_{\partial D(0,r)} |\log |\tilde{s}_N(u)|| d\sigma_r(u_1) \cdots d\sigma_r(u_m) \leq \frac{CN}{\delta^m}$ for some constant C outside an event of probability at most $e^{-e^N} + e^{-Q_{\kappa r,m}(N)}$.

Proof. Applying Lemma 4.6 to $U = (D(0,r))^m$, we have

$$\gamma_N \left\{ \sup_{u \in (\partial D(0,r))^m} |\tilde{s}_N(u)| > (1 + mr^2)^{\frac{N}{2}} e^{\eta N} \right\} \leq \gamma_N \left\{ \sup_{u \in (\bar{D}(0,r))^m} |\tilde{s}_N(u)| > (1 + mr^2)^{\frac{N}{2}} e^{\eta N} \right\} \leq e^{-e^{\eta N}}. \quad (4.20)$$

Therefore, take $\eta = 1$, outside an event of probability at most e^{-e^N} ,

$$\begin{aligned} \log^+ |\tilde{s}_N(u)| &\leq \frac{1}{2}N \log(1 + mr^2) + N \text{ on } (\partial D(0, r))^m, \\ \Rightarrow \int_{\partial D(0, r)} \cdots \int_{\partial D(0, r)} \log^+ |\tilde{s}_N(u)| d\sigma_r(u_1) \cdots d\sigma_r(u_m) &\leq \frac{1}{2}N \log(1 + mr^2) + N. \end{aligned} \quad (4.21)$$

Applying Lemma 4.5 to the distinguished boundary $(\partial D(0, \kappa r))^m$, we have: outside an event of probability at most $e^{-Q_{\kappa r, m}(N)}$, $\sup_{u \in (\partial D(0, \kappa r))^m} |\tilde{s}_N(u)| \geq 1$, i.e. $\exists \eta \in (\partial D(0, \kappa r))^m$ such that $|\tilde{s}_N(\eta)| \geq 1$,

$$\begin{aligned} 0 \leq \log |\tilde{s}_N(\eta)| &\leq \int_{\partial D(0, r)} \cdots \int_{\partial D(0, r)} \log |\tilde{s}_N(u)| \prod_{i=1}^m P_r(\eta_i, u_i) d\sigma_r(u_1) \cdots d\sigma_r(u_m) \\ &= \int_{\partial D(0, r)} \cdots \int_{\partial D(0, r)} \log^+ |\tilde{s}_N(u)| \prod_{i=1}^m P_r(\eta_i, u_i) d\sigma_r(u_1) \cdots d\sigma_r(u_m) \\ &\quad - \int_{\partial D(0, r)} \cdots \int_{\partial D(0, r)} \log^- |\tilde{s}_N(u)| \prod_{i=1}^m P_r(\eta_i, u_i) d\sigma_r(u_1) \cdots d\sigma_r(u_m), \end{aligned} \quad (4.22)$$

Since $\forall 1 \leq i \leq m$, $|\eta_i| = \kappa r = (1 - \sqrt{\delta})r$ and $|u_i| = r$, $\frac{\sqrt{\delta}}{2} \leq P_r(\eta_i, u_i) \leq \frac{2}{\sqrt{\delta}}$, (4.22) implies that outside an event of probability at most $e^{-Q_{\kappa r, m}(N)}$,

$$\begin{aligned} &\left(\frac{\sqrt{\delta}}{2}\right)^m \int_{\partial D(0, r)} \cdots \int_{\partial D(0, r)} \log^- |\tilde{s}_N(u)| d\sigma_r(u_1) \cdots d\sigma_r(u_m) \\ &\leq \left(\frac{2}{\sqrt{\delta}}\right)^m \int_{\partial D(0, r)} \cdots \int_{\partial D(0, r)} \log^+ |\tilde{s}_N(u)| d\sigma_r(u_1) \cdots d\sigma_r(u_m). \end{aligned} \quad (4.23)$$

Combine (4.21) and (4.23), we get: outside an event of probability at most $e^{-e^N} + e^{-Q_{\kappa r, m}(N)}$,

$$\begin{aligned} &\int_{\partial D(0, r)} \cdots \int_{\partial D(0, r)} |\log |\tilde{s}_N(u)|| d\sigma_r(u_1) \cdots d\sigma_r(u_m) \\ &= \int_{\partial D(0, r)} \cdots \int_{\partial D(0, r)} \log^+ |\tilde{s}_N(u)| d\sigma_r(u_1) \cdots d\sigma_r(u_m) + \int_{\partial D(0, r)} \cdots \int_{\partial D(0, r)} \log^- |\tilde{s}_N(u)| d\sigma_r(u_1) \cdots d\sigma_r(u_m) \\ &\leq \left[1 + \left(\frac{4}{\delta}\right)^m\right] \int_{\partial D(0, r)} \cdots \int_{\partial D(0, r)} \log^+ |\tilde{s}_N(u)| d\sigma_r(u_1) \cdots d\sigma_r(u_m) \\ &\leq \left[1 + \left(\frac{4}{\delta}\right)^m\right] \left[\frac{1}{2}N \log(1 + mr^2) + N\right] = \frac{CN}{\delta^m}. \end{aligned}$$

□

The following lemma estimates $\max_{u \in (\partial D(0, r))^m} \left| \sum_{J \in \Gamma_{m, N}} \prod_{i=1}^m \frac{P_r(\xi_{i, j_i}, u_i)}{N+1} - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d_m x \right|$:

Lemma 4.8. $\max_{u \in (\partial D(0, r))^m} \left| \sum_{J \in \Gamma_{m, N}} \prod_{i=1}^m \frac{P_r(\xi_{i, j_i}, u_i)}{N+1} - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d_m x \right| \leq \frac{o(1)}{\delta^{\frac{1}{2}(m+1)}}.$

Proof. For all $u \in (\partial D(0, r))^m$,

$$\begin{aligned} &\left| \sum_{J \in \Gamma_{m, N}} \prod_{i=1}^m \frac{P_r(\xi_{i, j_i}, u_i)}{N+1} - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d_m x \right| \\ &= \left| \sum_{\tau(J) \in \Gamma_{m, N}} \prod_{i=1}^m \frac{P_r(\xi_{i, \tau(j_i)}, u_i)}{N+1} - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d_m x \right| \\ &\leq \sum_{0 \leq t_1, \dots, t_m \leq p-1} \left| \sum_{J \in I_{t_1} \times \cdots \times I_{t_m}: \tau(J) \in \Gamma_{m, N}} \prod_{i=1}^m \frac{P_r(z_{j_i}, u_i)}{N+1} - \int_{H_{t_1, \dots, t_m}} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d_m x \right| \\ &\leq \sum_{0 \leq t_1, \dots, t_m \leq p-1} \left| \sum_{J \in I_{t_1} \times \cdots \times I_{t_m}: \tau(J) \in \Gamma_{m, N}} \prod_{i=1}^m \frac{P_r(z_{j_i}, u_i)}{N+1} - \int_{H_{t_1, \dots, t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d_m x \right| \\ &\quad + \sum_{0 \leq t_1, \dots, t_m \leq p-1} \left| \int_{H_{t_1, \dots, t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d_m x - \int_{H_{t_1, \dots, t_m}} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d_m x \right|, \end{aligned} \quad (4.24)$$

where $H_{t_1, \dots, t_m}(N) = \bigcup_{J \in I_{t_1} \times \dots \times I_{t_m}: \tau(J) \in \Gamma_{m,N}} \left[\frac{j_1}{N+1}, \frac{j_1+1}{N+1} \right] \times \dots \times \left[\frac{j_m}{N+1}, \frac{j_m+1}{N+1} \right]$.
 $\forall 0 \leq t_1, \dots, t_m \leq p-1$,

$$\begin{aligned}
& \left| \sum_{J \in I_{t_1} \times \dots \times I_{t_m}: \tau(J) \in \Gamma_{m,N}} \prod_{i=1}^m \frac{P_r(z_{j_i}, u_i)}{N+1} - \int_{H_{t_1, \dots, t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d_m x \right| \\
& \leq \sum_{J \in I_{t_1} \times \dots \times I_{t_m}: \tau(J) \in \Gamma_{m,N}} \int_{\left[\frac{j_1}{N+1}, \frac{j_1+1}{N+1} \right] \times \dots \times \left[\frac{j_m}{N+1}, \frac{j_m+1}{N+1} \right]} \left| \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) - \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}\frac{j_i}{N+1}}, u_i) \right| d_m x \\
& \leq \frac{(q+1)^m}{(N+1)^m} m \sup_{|\omega|=\kappa r, |u|=r} [P_r(\omega, u)]^{m-1} \sup_{|\omega| \leq \kappa r, |u|=r} \left| \frac{\partial P_r(\omega, u)}{\partial \omega} \right| \frac{2\pi\kappa r}{N+1} \\
& \leq \frac{C}{p^m \delta^{\frac{1}{2}(m+1)} (N+1)}, \\
& \Rightarrow \sum_{0 \leq t_1, \dots, t_m \leq p-1} \left| \sum_{J \in I_{t_1} \times \dots \times I_{t_m}: \tau(J) \in \Gamma_{m,N}} \prod_{i=1}^m \frac{P_r(z_{j_i}, u_i)}{N+1} - \int_{H_{t_1, \dots, t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d_m x \right| \\
& \leq \frac{C}{\delta^{\frac{1}{2}(m+1)} (N+1)} = \frac{o(1)}{\delta^{\frac{1}{2}(m+1)}} \tag{4.25}
\end{aligned}$$

To bound the second term in (4.24), we need the following statement, which can be proved in a similar way as Lemma 4.3:

$$\lim_{N \rightarrow \infty} \text{Vol}_{\mathbb{R}^m}(H_{t_1, \dots, t_m}(N) \triangle H_{t_1, \dots, t_m}) = 0 \text{ for any } 0 \leq t_1, \dots, t_m \leq p-1.$$

Hence,

$$\begin{aligned}
& \sum_{0 \leq t_1, \dots, t_m \leq p-1} \left| \int_{H_{t_1, \dots, t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d_m x - \int_{H_{t_1, \dots, t_m}} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d_m x \right| \\
& \leq \sum_{0 \leq t_1, \dots, t_m \leq p-1} \text{Vol}_{\mathbb{R}^m}(H_{t_1, \dots, t_m}(N) \triangle H_{t_1, \dots, t_m}) \left[\sup_{|\omega|=\kappa r, |u|=r} P_r(\omega, u) \right]^m \\
& \leq \sum_{0 \leq t_1, \dots, t_m \leq p-1} o(1) \left(\frac{2}{\sqrt{\delta}} \right)^m \\
& = \frac{o(1)}{\delta^{\frac{1}{2}m}}. \tag{4.26}
\end{aligned}$$

This $o(1)$ may depend on p .

By (4.24), (4.25) and (4.26), we prove the lemma. \square

Combine (4.19), Lemma 4.7 and Lemma 4.8, we have: outside an event of probability at most $e^{-e^N} + e^{-Q_{\kappa r, m}(N)}$,

$$I \leq (N+1)^m \frac{o(1)}{\delta^{\frac{1}{2}(m+1)}} \frac{CN}{\delta^m} = \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m + \frac{1}{2}}}.$$

By changing the order of integration,

$$II = (N+1)^m \int_H \int_{\partial D(0, r)} \dots \int_{\partial D(0, r)} \log |\tilde{s}_N(u)| \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d\sigma_r(u_1) \dots d\sigma_r(u_m) d_m x.$$

If \tilde{s}_N is nonvanishing on $(\bar{D}(0, r))^m$, $\log |\tilde{s}_N(u)|$ is harmonic in $u_i \in$ a neighbourhood of $\bar{D}(0, r)$ for each fixed $(u_1, \dots, \hat{u}_i, \dots, u_m) \in (\bar{D}(0, r))^{m-1}$. Applying mean value theorem for harmonic functions, we get

$$\begin{aligned} II &= (N+1)^m \times \\ &\quad \int_H \int_{\partial D(0, r)} \cdots \int_{\partial D(0, r)} \log |\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, u_2, \dots, u_m)| \prod_{i=2}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i) d\sigma_r(u_2) \cdots d\sigma_r(u_m) d_m x \\ &= \dots \\ &= (N+1)^m \int_H \log |\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, \dots, \kappa r e^{2\pi\sqrt{-1}x_m})| d_m x. \end{aligned}$$

Denote

$$\Xi = \int_H \log |\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, \dots, \kappa r e^{2\pi\sqrt{-1}x_m})| d_m x, \quad (4.27)$$

which is a complex random variable. Thus we have proved:

Lemma 4.9. *If \tilde{s}_N is nonvanishing on $(\bar{D}(0, r))^m$, then outside an event of probability at most $e^{-e^N} + e^{-Q_{\kappa r, m}(N)}$,*

$$\log \prod_{J \in \Gamma_{m, N}} |\zeta_J| \leq \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m + \frac{1}{2}}} + (N+1)^m \Xi.$$

Replace $\Gamma_{m, N} = \{J = (j_1, \dots, j_m) \in [0, N]^m \cap \mathbb{Z}^m : 0 \leq j_1 \leq \dots \leq j_m \leq N\}$ by $\Gamma_{m, N}^{(\varrho)} = \{J = (j_1, \dots, j_m) \in [0, N]^m \cap \mathbb{Z}^m : 0 \leq j_{\varrho(1)} \leq \dots \leq j_{\varrho(m)} \leq N\}$, where ϱ can be any element in S_m , the permutation group of m letters. Then similar results hold and we have counterparts for Lemma 4.4 and Lemma 4.9, which we state without proof.

Lemma 4.10. *Denote the covariance matrix of the random vector $(\zeta_J^{(\varrho)} = \tilde{s}_N(\xi_J))_{J \in \Gamma_{m, N}^{(\varrho)}}$ by $\Sigma^{(\varrho)}$. Then $\log(\det \Sigma^{(\varrho)}) = Q_{\kappa r, m}(N) + \frac{2\beta_m}{p} N^{m+1} + o(N^{m+1})$.*

$\forall \varrho \in S_m$, denote

$$\begin{aligned} H^{(\varrho)} &= \bigcup_{0 \leq t_1, \dots, t_m \leq p-1} H_{t_1, \dots, t_m}^{(\varrho)} \\ &:= \bigcup_{0 \leq t_1, \dots, t_m \leq p-1} \left\{ x = (x_1, \dots, x_m) \in \mathbb{R}^m : 0 \leq x_{\varrho(1)} - \frac{t_{\varrho(1)}}{p} \leq \dots \leq x_{\varrho(m)} - \frac{t_{\varrho(m)}}{p} \leq \frac{1}{p} \right\} \end{aligned}$$

and the random variable

$$\Xi^{(\varrho)} = \int_{H^{(\varrho)}} \log |\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, \dots, \kappa r e^{2\pi\sqrt{-1}x_m})| d_m x.$$

Then

Lemma 4.11. *If \tilde{s}_N is nonvanishing on $(\bar{D}(0, r))^m$, then outside an event of probability at most $e^{-e^N} + e^{-Q_{\kappa r, m}(N)}$,*

$$\log \prod_{J \in \Gamma_{m, N}^{(\varrho)}} |\zeta_J^{(\varrho)}| \leq \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m + \frac{1}{2}}} + (N+1)^m \Xi^{(\varrho)}.$$

If \tilde{s}_N is nonvanishing on $(\bar{D}(0, r))^m$,

$$\begin{aligned}
\sum_{\varrho \in S_m} \Xi^{(\varrho)} &= \sum_{\varrho \in S_m} \int_{H^{(\varrho)}} \log |\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, \dots, \kappa r e^{2\pi\sqrt{-1}x_m})| d_m x \\
&= \int_{\bigcup_{\varrho \in S_m} H^{(\varrho)}} \log |\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, \dots, \kappa r e^{2\pi\sqrt{-1}x_m})| d_m x \\
&= \int_0^1 \dots \int_0^1 \log |\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, \dots, \kappa r e^{2\pi\sqrt{-1}x_m})| dx_1 \dots dx_m \\
&= \int_{\partial D(0, \kappa r)} \dots \int_{\partial D(0, \kappa r)} \log |\tilde{s}_N(\omega_1, \dots, \omega_m)| d\sigma_{\kappa r}(\omega_1) \dots d\sigma_{\kappa r}(\omega_m) \\
&= \log |\tilde{s}_N(0, \dots, 0)| \\
&= \log |c_{(0, \dots, 0)}|,
\end{aligned}$$

the second equality holds because for distinct $\varrho_1, \varrho_2 \in S_m$, $H^{(\varrho_1)} \cap H^{(\varrho_2)}$ is of m -dimensional Lebesgue measure zero.

Proof of the upper bound in Theorem 0.1.

$$\begin{aligned}
P_{0,m}(r, N) &= \gamma_N \{0 \notin \tilde{s}_N((\bar{D}(0, r))^m)\} \\
&= \gamma_N \{(\log |c_{(0, \dots, 0)}| > 2m! \log N) \cap (0 \notin \tilde{s}_N((\bar{D}(0, r))^m))\} \\
&\quad + \gamma_N \{(\log |c_{(0, \dots, 0)}| \leq 2m! \log N) \cap (0 \notin \tilde{s}_N((\bar{D}(0, r))^m))\} \\
&\leq \gamma_N (|c_{(0, \dots, 0)}| > N^{2m!}) + \gamma_N \{(\sum_{\varrho \in S_m} \Xi^{(\varrho)} \leq 2m! \log N) \cap (0 \notin \tilde{s}_N((\bar{D}(0, r))^m))\} \\
&\leq e^{-N^{4m!}} + \gamma_N \left\{ \bigcup_{\varrho \in S_m} (\Xi^{(\varrho)} \leq 2 \log N) \cap (0 \notin \tilde{s}_N((\bar{D}(0, r))^m)) \right\} \\
&\leq e^{-N^{4m!}} + \sum_{\varrho \in S_m} \gamma_N \{(\Xi^{(\varrho)} \leq 2 \log N) \cap (0 \notin \tilde{s}_N((\bar{D}(0, r))^m))\}.
\end{aligned}$$

Lemma 4.9 implies

$$\begin{aligned}
&\gamma_N \{(\Xi \leq 2 \log N) \cap (0 \notin \tilde{s}_N((\bar{D}(0, r))^m))\} \\
&\leq e^{-e^N} + e^{-Q_{\kappa r, m}(N)} + \gamma_N \left\{ \log \prod_{J \in \Gamma_{m, N}} |\zeta_J| \leq \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m + \frac{1}{2}}} + 2(N+1)^m \log N \right\} \\
&= e^{-e^N} + e^{-Q_{\kappa r, m}(N)} + \gamma_N \left\{ \prod_{J \in \Gamma_{m, N}} |\zeta_J| \leq \exp \left\{ \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m + \frac{1}{2}}} + 2(N+1)^m \log N \right\} \right\}.
\end{aligned}$$

Denote

$$\mathcal{E}_{m, N} = \left\{ \zeta = (\zeta_J)_{J \in \Gamma_{m, N}} \in \mathbb{C}^{\binom{N+m}{m}} : \prod_{J \in \Gamma_{m, N}} |\zeta_J| \leq \exp \left\{ \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m + \frac{1}{2}}} + 2(N+1)^m \log N \right\} \right\},$$

and

$$\mathcal{F}_{m, N} = \left\{ \zeta = (\zeta_J)_{J \in \Gamma_{m, N}} \in \mathcal{E}_{m, N} : |\zeta_J| \leq (2 + 2mr^2)^{\frac{N}{2}}, \forall J \in \Gamma_{m, N} \right\} \subset \mathcal{E}_{m, N},$$

both of which can be treated as subsets in $\mathbb{C}^{\binom{N+m}{m}}$ and events in the probability space $(H^0(\mathbb{CP}^m, \mathcal{O}(N)), \gamma_N)$. Thus,

$$\begin{aligned}
\gamma_N \{(\Xi \leq 2 \log N) \cap (0 \notin \tilde{s}_N((\bar{D}(0, r))^m))\} &\leq e^{-e^N} + e^{-Q_{\kappa r, m}(N)} + \gamma_N(\mathcal{E}_{m, N}) \\
&\leq e^{-e^N} + e^{-Q_{\kappa r, m}(N)} + \gamma_N(\mathcal{E}_{m, N} \setminus \mathcal{F}_{m, N}) + \gamma_N(\mathcal{F}_{m, N}).
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
\gamma_N(\mathcal{E}_{m,N} \setminus \mathcal{F}_{m,N}) &\leq \gamma_N\{|\zeta_J| > (2+2mr^2)^{\frac{N}{2}} \text{ for some } J \in \Gamma_{m,N}\} \\
&\leq \gamma_N\left\{\sup_{\omega \in (\partial D(0, \kappa r))^m} |\tilde{s}_N(\omega)| > (2+2mr^2)^{\frac{N}{2}}\right\} \\
&\leq \gamma_N\left\{\sup_{\omega \in (\bar{D}(0, r))^m} |\tilde{s}_N(\omega)| > (1+mr^2)^{\frac{N}{2}} 2^{\frac{N}{2}}\right\} \\
&\leq e^{-2^{\frac{N}{2}}},
\end{aligned} \tag{4.29}$$

where the last inequality is due to Lemma 4.6.

$$\begin{aligned}
\gamma_N(\mathcal{F}_{m,N}) &= \frac{1}{\pi \binom{N+m}{m} \det \Sigma} \int_{\mathcal{F}_{m,N}} e^{-\zeta^* \Sigma^{-1} \zeta} d_{2 \binom{N+m}{m}} \zeta \\
&\leq \exp\left\{-[Q_{\kappa r, m}(N) + \frac{2\beta_m}{p} N^{m+1}] + o(N^{m+1})\right\} \pi^{-\binom{N+m}{m}} \text{Vol}_{\mathbb{C}^{\binom{N+m}{m}}}(\mathcal{F}_{m,N})
\end{aligned}$$

by Lemma 4.4. Change into polar coordinates and denote

$$\begin{aligned}
&\text{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathcal{F}_{m,N}) \\
&= \text{Vol}_{\mathbb{R}^{\binom{N+m}{m}}} \left\{ (x_J)_{J \in \Gamma_{m,N}} \in [0, (2+2mr^2)^{\frac{N}{2}}]^{\binom{N+m}{m}} : \prod_{J \in \Gamma_{m,N}} x_J \leq \exp\left\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + 2(N+1)^m \log N\right\} \right\}, \\
&\Rightarrow \gamma_N(\mathcal{F}_{m,N}) \\
&\leq 2^{\binom{N+m}{m}} \exp\left\{-[Q_{\kappa r, m}(N) + \frac{2\beta_m}{p} N^{m+1}] + o(N^{m+1})\right\} \exp\left\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + 2(N+1)^m \log N\right\} \text{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathcal{F}_{m,N}) \\
&= 2^{\binom{N+m}{m}} \exp\left\{-[Q_{\kappa r, m}(N) + \frac{2\beta_m}{p} N^{m+1}] + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\right\} \text{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathcal{F}_{m,N}).
\end{aligned}$$

Since $\binom{N+m}{m} \frac{N}{2} \log(2+2mr^2) - \left[\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + 2(N+1)^m \log N\right] > \binom{N+m}{m}$ for N large (up to now p, δ are constants), we can apply Lemma 4.6 in [4] and get:

$$\begin{aligned}
&\text{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathcal{F}_{m,N}) \\
&\leq \frac{\exp\left\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + 2(N+1)^m \log N\right\}}{[(\binom{N+m}{m} - 1)!]} \left\{ \binom{N+m}{m} \frac{N}{2} \log(2+2mr^2) - \left[\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + 2(N+1)^m \log N\right] \right\}^{\binom{N+m}{m}} \\
&\leq \frac{\exp\left\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + 2(N+1)^m \log N\right\}}{2^{\binom{N+m}{m}} [(\binom{N+m}{m} - 1)!]} \left[N \binom{N+m}{m} \log(2+2mr^2) \right]^{\binom{N+m}{m}} \\
&\Rightarrow \gamma_N(\mathcal{F}_{m,N}) \leq \frac{\exp\left\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + 2(N+1)^m \log N - [Q_{\kappa r, m}(N) + \frac{2\beta_m}{p} N^{m+1}]\right\}}{[(\binom{N+m}{m} - 1)!]} \\
&\quad \times \left[N \binom{N+m}{m} \log(2+2mr^2) \right]^{\binom{N+m}{m}}, \\
&\Rightarrow \log \gamma_N(\mathcal{F}_{m,N}) \leq \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + 2(N+1)^m \log N - [Q_{\kappa r, m}(N) + \frac{2\beta_m}{p} N^{m+1}] \\
&\quad + \binom{N+m}{m} \log \left[N \binom{N+m}{m} \log(2+2mr^2) \right] - \log [(\binom{N+m}{m} - 1)!] \\
&= -Q_{\kappa r, m}(N) - \frac{2\beta_m}{p} N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}.
\end{aligned} \tag{4.30}$$

By Lemma 2.2, (4.28), (4.29) and (4.30),

$$\begin{aligned}
& \gamma_N \left\{ (\Xi \leq 2 \log N) \cap \left(0 \notin \tilde{s}_N((\bar{D}(0, r))^m) \right) \right\} \\
& \leq e^{-e^N} + e^{-Q_{\kappa r, m}(N)} + e^{-2 \frac{N}{2}} + \exp \left\{ -Q_{\kappa r, m}(N) - \frac{2\beta_m}{p} N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m + \frac{1}{2}}} \right\} \\
& \leq \exp \left\{ -\min \left\{ \frac{2m \log(\kappa r)}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k}, \frac{2m \log(\kappa r)}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k} + \frac{2\beta_m}{p} \right\} N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m + \frac{1}{2}}} \right\}.
\end{aligned}$$

Similarly, $\forall \varrho \in S_m$,

$$\begin{aligned}
& \gamma_N \left\{ (\Xi^{(\varrho)} \leq 2 \log N) \cap \left(0 \notin \tilde{s}_N((\bar{D}(0, r))^m) \right) \right\} \\
& \leq \exp \left\{ -\min \left\{ \frac{2m \log(\kappa r)}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k}, \frac{2m \log(\kappa r)}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k} + \frac{2\beta_m}{p} \right\} N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m + \frac{1}{2}}} \right\},
\end{aligned}$$

$$\begin{aligned}
& \Rightarrow P_{0, m}(r, N) \\
& \leq e^{-N^{4m!}} + m! \exp \left\{ -\min \left\{ \frac{2m \log(\kappa r)}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k}, \frac{2m \log(\kappa r)}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k} + \frac{2\beta_m}{p} \right\} N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m + \frac{1}{2}}} \right\} \\
& = \exp \left\{ -\min \left\{ \frac{2m \log(\kappa r)}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k}, \frac{2m \log(\kappa r)}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k} + \frac{2\beta_m}{p} \right\} N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m + \frac{1}{2}}} \right\}, \\
& \Rightarrow \log P_{0, m}(r, N) \leq -\min \left\{ \frac{2m \log(\kappa r)}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k}, \frac{2m \log(\kappa r)}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k} + \frac{2\beta_m}{p} \right\} N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m + \frac{1}{2}}}, \\
& \Rightarrow \limsup_{N \rightarrow \infty} \frac{\log P_{0, m}(r, N)}{N^{m+1}} \leq -\min \left\{ \frac{2m \log(\kappa r)}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k}, \frac{2m \log(\kappa r)}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k} + \frac{2\beta_m}{p} \right\}.
\end{aligned}$$

Let $p \rightarrow \infty$, then

$$\Rightarrow \limsup_{N \rightarrow \infty} \frac{\log P_{0, m}(r, N)}{N^{m+1}} \leq -\left[\frac{2m \log(\kappa r)}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k} \right].$$

Let $\delta \rightarrow 0+$, then $\kappa = 1 - \sqrt{\delta} \rightarrow 1$,

$$\begin{aligned}
& \Rightarrow \limsup_{N \rightarrow \infty} \frac{\log P_{0, m}(r, N)}{N^{m+1}} \leq -\left[\frac{2m \log r}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k} \right]. \\
& \Rightarrow \log P_{0, m}(r, N) \leq -\left[\frac{2m \log r}{(m+1)!} + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k} \right] N^{m+1} + o(N^{m+1}).
\end{aligned}$$

□

5. PROOF OF THEOREM 0.2

5.1. Lower bound.

Definition 5.1.

$$\begin{aligned}
\Lambda_{m, N}(r) &:= \left\{ K \in \Lambda_{m, N} : \binom{N}{K} r^{2|K|} \geq 1 \right\} \subset \Lambda_{m, N}, \\
R_{r, m}(N) &:= \sum_{K \in \Lambda_{m, N}(r)} \log \left[\binom{N}{K} r^{2|K|} \right].
\end{aligned}$$

Lemma 5.2. $\log P_{0, m}(r, N) \geq -R_{r, m}(N) + o(N^{m+1})$.

Proof. Consider the following event $\Omega_{r,m,N}$:

$$\begin{aligned} (i) \quad & |c_{(0,\dots,0)}| \geq \sqrt{N}, \\ (ii) \quad & |c_K| \leq \frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}r^{|K|}\binom{|K|+m-1}{m-1}}}, \quad K \in \Lambda_{m,N}(r) \setminus \{(0,\dots,0)\}, \\ (iii) \quad & |c_K| \leq \frac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}}, \quad K \in \Lambda_{m,N} \setminus \Lambda_{m,N}(r). \end{aligned}$$

Then when $\Omega_{r,m,N}$ occurs, $\forall z \in (\bar{D}(0,r))^m$,

$$\begin{aligned} |\tilde{s}_N(z)| &\geq \sqrt{N} - \sum_{K \in \Lambda_{m,N}(r) \setminus \{(0,\dots,0)\}} \frac{\sqrt{\binom{N}{K}r^{|K|}}}{2\sqrt{N}\sqrt{\binom{N}{K}r^{|K|}\binom{|K|+m-1}{m-1}}} - \sum_{K \in \Lambda_{m,N} \setminus \Lambda_{m,N}(r)} \frac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}} \\ &= \sqrt{N} - \sum_{K \in \Lambda_{m,N} \setminus \{(0,\dots,0)\}} \frac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}} \\ &= \sqrt{N} - \sum_{k=1}^N \frac{1}{2\sqrt{N}} \\ &= \frac{1}{2}\sqrt{N} > 0. \end{aligned}$$

Thus,

$$\begin{aligned} P_{0,m}(r,N) &\geq \gamma_N(\Omega_{r,m,N}) \\ &= \gamma_N(|c_{(0,\dots,0)}| \geq \sqrt{N}) \prod_{K \in \Lambda_{m,N}(r) \setminus \{(0,\dots,0)\}} \gamma_N\left(|c_K| \leq \frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}r^{|K|}\binom{|K|+m-1}{m-1}}}\right) \\ &\quad \times \prod_{K \in \Lambda_{m,N} \setminus \Lambda_{m,N}(r)} \gamma_N\left(|c_K| \leq \frac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}}\right) \\ &\geq e^{-N} \prod_{K \in \Lambda_{m,N}(r) \setminus \{(0,\dots,0)\}} \frac{1}{8N\binom{N}{K}r^{2|K|}\binom{|K|+m-1}{m-1}^2} \prod_{K \in \Lambda_{m,N} \setminus \Lambda_{m,N}(r)} \frac{1}{8N\binom{|K|+m-1}{m-1}^2}, \\ &\Rightarrow \log P_{0,m}(r,N) \geq -N - \sum_{K \in \Lambda_{m,N}(r) \setminus \{(0,\dots,0)\}} \log\left[\binom{N}{K}r^{2|K|}\right] - \sum_{K \in \Lambda_{m,N}(r) \setminus \{(0,\dots,0)\}} \log\left[8N\binom{|K|+m-1}{m-1}^2\right] \\ &= - \sum_{K \in \Lambda_{m,N}(r) \setminus \{(0,\dots,0)\}} \log\left[\binom{N}{K}r^{2|K|}\right] + o(N^{m+1}) \\ &= -R_{r,m}(N) + o(N^{m+1}). \end{aligned}$$

□

5.2. Upper bound. For some $\alpha \in (0, 1]$, we can define the index sets $\Lambda_{m, \lfloor \alpha N \rfloor}$, $\Gamma_{m, \lfloor \alpha N \rfloor}$ and the $\binom{\lfloor \alpha N \rfloor + m}{m} \times \binom{\lfloor \alpha N \rfloor + m}{m}$ matrix

$$W_{m, \lfloor \alpha N \rfloor}(\xi) = (\xi_J^K)_{J \in \Gamma_{m, \lfloor \alpha N \rfloor}, K \in \Lambda_{m, \lfloor \alpha N \rfloor}}.$$

We also assign the values of the variables $(\xi_{i,j})_{0 \leq i \leq m, 0 \leq j \leq \lfloor \alpha N \rfloor}$ to be the points on $\partial D(0, \kappa r)$ in a way similar to Section 4 except that we replace N by $\lfloor \alpha N \rfloor$. Then we have the following lemma.

Lemma 5.3.

$$\log |\det W_{m, \lfloor \alpha N \rfloor}(\xi)| = m \binom{\lfloor \alpha N \rfloor + m}{m+1} \log(\kappa r) + \frac{\beta_m}{p} (\lfloor \alpha N \rfloor)^{m+1} + o(N^{m+1}).$$

$\zeta = (\zeta_J)^t_{J \in \Gamma_{m, \lfloor \alpha N \rfloor}} = (\tilde{s}_N(\xi_J))^t_{J \in \Gamma_{m, \lfloor \alpha N \rfloor}}$ is a dimension $\binom{\lfloor \alpha N \rfloor + m}{m}$ mean zero complex Gaussian random vector with covariance matrix

$$\Sigma = V_{m,N,\alpha}(\xi) V_{m,N,\alpha}^*(\xi),$$

where $V_{m,N,\alpha}(\xi) = \left(\sqrt{\binom{N}{K}} \xi_J^K \right)_{J \in \Gamma_{m, \lfloor \alpha N \rfloor}, K \in \Lambda_{m,N}}$ is an $\binom{\lfloor \alpha N \rfloor + m}{m} \times \binom{N+m}{m}$ matrix.

Definition 5.4. $Q_{r,m,\alpha}(N) := \sum_{K \in \Lambda_{m, \lfloor \alpha N \rfloor}} \log \left[\binom{N}{K} r^{2|K|} \right].$

Lemma 5.5. $\log \det \Sigma \geq Q_{\kappa r, m, \alpha}(N) + \frac{2\beta_m}{p} (\lfloor \alpha N \rfloor)^{m+1} + o(N^{m+1}).$

Proof. By Cauchy-Binet identity,

$$\begin{aligned} \det \Sigma &= \sum_{M: \binom{\lfloor \alpha N \rfloor + m}{m} \times \binom{\lfloor \alpha N \rfloor + m}{m} \text{ minor of } V_{m,N,\alpha}(\xi)} |\det M|^2 \\ &\geq \left| \det \left(\sqrt{\binom{N}{K}} \xi_J^K \right)_{J \in \Gamma_{m, \lfloor \alpha N \rfloor}, K \in \Lambda_{m, \lfloor \alpha N \rfloor}} \right|^2 \\ &= \prod_{K \in \Lambda_{m, \lfloor \alpha N \rfloor}} \binom{N}{K} |\det W_{m, \lfloor \alpha N \rfloor}(\xi)|^2 \\ \Rightarrow \log \det \Sigma &\geq \sum_{K \in \Lambda_{m, \lfloor \alpha N \rfloor}} \log \binom{N}{K} + 2m \binom{\lfloor \alpha N \rfloor + m}{m+1} \log(\kappa r) + \frac{2\beta_m}{p} (\lfloor \alpha N \rfloor)^{m+1} + o(N^{m+1}) \\ &= \sum_{K \in \Lambda_{m, \lfloor \alpha N \rfloor}} \log \left[\binom{N}{K} (\kappa r)^{2|K|} \right] + \frac{2\beta_m}{p} (\lfloor \alpha N \rfloor)^{m+1} + o(N^{m+1}) \\ &= Q_{\kappa r, m, \alpha}(N) + \frac{2\beta_m}{p} (\lfloor \alpha N \rfloor)^{m+1} + o(N^{m+1}). \end{aligned}$$

□

The following lemma is a counterpart of Lemma 4.9. The proof is similar.

Lemma 5.6. *If \tilde{s}_N is nonvanishing on $(\bar{D}(0, r))^m$, then outside an event of probability at most $e^{-e^N} + e^{-R_{\kappa r, m}(N)}$,*

$$\log \prod_{J \in \Gamma_{m, \lfloor \alpha N \rfloor}} |\zeta_J| \leq \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m + \frac{1}{2}}} + (\lfloor \alpha N \rfloor + 1)^m \Xi,$$

where the complex random variable Ξ is defined in (4.27).

By playing the same trick of permutation as in Section 4, we can get an upper bound estimate for $P_{0,m}(r, N)$:

$$P_{0,m}(r, N) \leq e^{-N^{4m!}} + m! \left\{ e^{-e^N} + e^{-R_{\kappa r, m}(N)} + e^{-2\frac{N}{\delta}} + \exp \left[-Q_{\kappa r, m, \alpha}(N) - \frac{2\beta_m}{p} (\lfloor \alpha N \rfloor)^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m + \frac{1}{2}}} \right] \right\}. \quad (5.1)$$

5.3. Punch line of the proof. In order to prove Theorem 0.2, it suffices to compute $R_{r,m}(N)$ and $Q_{r,m,\alpha}(N)$ asymptotically. We follow the same idea in Lemma 2.2.

The scaled lattice $\frac{1}{N} \Lambda_{m,N}(r)$ corresponds to the set

$$\{x = (x_1, \dots, x_m) \in \Sigma_m : E_r(x) \geq 0\}$$

and $\frac{1}{N} \Lambda_{r,m,\alpha}(N)$ corresponds to the set

$$\{x = (x_1, \dots, x_m) \in \mathbb{R}^{m+} : \sum_{i=1}^m x_i \leq \alpha \leq 1\}.$$

So we have

$$R_{r,m}(N) = \sum_{K \in \Lambda_{m,N}(r)} \log \left[\binom{N}{K} r^{2|K|} \right] = N^{m+1} \int_{x \in \Sigma_m : E_r(x) \geq 0} E_r(x) d_m x + o(N^{m+1}), \quad (5.2)$$

$$Q_{r,m,\alpha}(N) = \sum_{K \in \Lambda_{m, \lfloor \alpha N \rfloor}} \log \left[\binom{N}{K} r^{2|K|} \right] = N^{m+1} \int_{x \in \mathbb{R}^{m+1}: \sum_{i=1}^m x_i \leq \alpha} E_r(x) d_m x + o(N^{m+1}). \quad (5.3)$$

Moreover, if we go through the proof of Lemma 2.2, we find that the $o(N^{m+1})$ terms in (5.2) and (5.3) are uniform if $r \leq c$ for some constant $c > 0$, which implies that when r is replaced by $\kappa r = (1 - \sqrt{\delta})r$, the remainder won't depend on δ .

Proof of Theorem 0.2. The lower bound proof is already implied by Lemma 5.2 and (5.2). To prove the upper bound, by (5.1) and (5.3),

$$\begin{aligned} & \log P_{0,m}(r, N) \\ & \leq -N^{m+1} \min \left\{ \int_{x \in \Sigma_m: E_{\kappa r}(x) \geq 0} E_{\kappa r}(x) d_m x, \int_{x \in \mathbb{R}^{m+1}: \sum_{i=1}^m x_i \leq \alpha} E_{\kappa r}(x) d_m x + \frac{2\beta_m \alpha^{m+1}}{p} \right\} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m + \frac{1}{2}}}. \end{aligned}$$

Similar as in Section 4, we can get

$$\begin{aligned} \log P_{0,m}(r, N) & \leq -N^{m+1} \min \left\{ \int_{x \in \Sigma_m: E_r(x) \geq 0} E_r(x) d_m x, \int_{x \in \mathbb{R}^{m+1}: \sum_{i=1}^m x_i \leq \alpha} E_r(x) d_m x \right\} + o(N^{m+1}) \\ & = -N^{m+1} \int_{x \in \mathbb{R}^{m+1}: \sum_{i=1}^m x_i \leq \alpha} E_r(x) d_m x + o(N^{m+1}). \end{aligned}$$

It amounts to find a proper $\alpha_0 = \alpha_0(r, m) \in (0, 1]$ which maximize $\int_{x \in \mathbb{R}^{m+1}: \sum_{i=1}^m x_i \leq \alpha} E_r(x) d_m x$. For this purpose we consider the function defined on $(0, 1]$

$$\Upsilon(\alpha) := \int_{x \in \mathbb{R}^{m+1}: \sum_{i=1}^m x_i \leq \alpha} E_r(x) d_m x.$$

Then

$$\begin{aligned} \Upsilon(\alpha) & = 2m \log r \int_{x \in \mathbb{R}^{m+1}: \sum_{i=1}^m x_i \leq \alpha} x_1 d_m x - m \int_{x \in \mathbb{R}^{m+1}: \sum_{i=1}^m x_i \leq \alpha} x_1 \log x_1 d_m x \\ & \quad - \int_{x \in \mathbb{R}^{m+1}: \sum_{i=1}^m x_i \leq \alpha} \left(1 - \sum_{i=1}^m x_i\right) \log \left(1 - \sum_{i=1}^m x_i\right) d_m x \\ & = 2m \log r \frac{\alpha^{m+1}}{(m+1)!} - m \frac{\alpha^{m+1}}{(m+1)!} \left[\log \alpha - \sum_{k=2}^{m+1} \frac{1}{k} \right] - \frac{1}{(m-1)!} \int_0^\alpha (1-x) x^{m-1} \log(1-x) dx, \\ \Upsilon'(\alpha) & = \frac{\alpha^{m-1}}{(m-1)!} \left\{ \left(2 \log r + \sum_{k=2}^m \frac{1}{k}\right) \alpha - [\alpha \log \alpha + (1-\alpha) \log(1-\alpha)] \right\}, \end{aligned}$$

where we take $\sum_{k=2}^m \frac{1}{k} = 0$ when $m = 1$. So if $2 \log r + \sum_{k=2}^m \frac{1}{k} \geq 0$, $\Upsilon'(\alpha) \geq 0$ over $(0, 1]$,

$$\max_{(0,1]} \Upsilon = \Upsilon(1), \Rightarrow \alpha_0 = 1.$$

If $2 \log r + \sum_{k=2}^m \frac{1}{k} < 0$, let $\alpha_0 \in (0, 1)$ be the nonzero root of $\left(2 \log r + \sum_{k=2}^m \frac{1}{k}\right) \alpha = \alpha \log \alpha + (1-\alpha) \log(1-\alpha)$,

$$\begin{aligned} \max_{(0,1]} \Upsilon & = \Upsilon(\alpha_0) = \int_{x \in \mathbb{R}^{m+1}: \sum_{i=1}^m x_i \leq \alpha_0} E_r(x) d_m x \\ & \quad \left(= \frac{1}{(m+1)!} \left[(1 - \alpha_0^m) \log(1 - \alpha_0) + \sum_{k=1}^m \frac{\alpha_0^k}{k} \right] \right). \end{aligned} \quad (5.4)$$

□

Remark 5.7. The proofs of Theorem 0.1 and 0.2 also work for a general polydisc $\prod_{i=1}^m D(0, r_i)$. For

example, if $r = (r_1, \dots, r_m) \in [1, \infty)^m$, the function E_r in Theorem 0.1 would be $E_r(x) = 2 \sum_{i=1}^m x_i \log r_i -$

$$\left[\sum_{i=1}^m x_i \log x_i + \left(1 - \sum_{i=1}^m x_i\right) \log \left(1 - \sum_{i=1}^m x_i\right) \right] \text{ and } \int_{\Sigma_m} E_r(x) d_m x = \frac{2}{(m+1)!} \sum_{i=1}^m \log r_i + \frac{1}{m!} \sum_{k=2}^{m+1} \frac{1}{k}.$$

6. HOLE PROBABILITY OF $SU(2)$ POLYNOMIALS

Proof of Corollary 0.4. When $r \geq 1$, $\alpha_0 = 1$. The result follows from Theorem 0.1. When $0 < r < 1$,

$$x \in \mathbb{R}^+ : E_r(x) = 2x \log r - [x \log x + (1-x) \log(1-x)] \geq 0 \Leftrightarrow 0 \leq x \leq \alpha_0.$$

By Theorem 0.2,

$$\log P_{0,1}(r, N) = -N^2 \int_0^{\alpha_0} E_r(x) dx + o(N^2),$$

where the value of the integral in the corollary is due to (5.4) and the fact that

$$2\alpha_0 \log r = \alpha_0 \log \alpha_0 + (1 - \alpha_0) \log(1 - \alpha_0).$$

□

Proof of Theorem 0.5. Since ∂U is a Jordan curve, by Carathéodory's theorem, ϕ can be extended to a homeomorphism $\bar{D}(0, 1) \rightarrow \bar{U}$. We still use ϕ to denote the extension map. Thus, $\tilde{s}_N(z) = \sum_{k=0}^N c_k \sqrt{\binom{N}{k}} z^k$ is nonvanishing over \bar{U} if and only if $t_N(\omega) := \sum_{k=0}^N c_k \sqrt{\binom{N}{k}} (\phi(\omega))^k$ is nonvanishing over $\bar{D}(0, 1)$, where $t_N \in \mathcal{O}(D(0, 1)) \cap \mathcal{C}(\bar{D}(0, 1))$. Since

$$\begin{bmatrix} t_N(0) \\ t'_N(0) \\ \vdots \\ t_N^{(N)}(0) \end{bmatrix} = A \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix},$$

where A is an $(N+1) \times (N+1)$ lower triangular matrix with diagonal entries $\{k! \sqrt{\binom{N}{k}} (\phi'(0))^k\}_{0 \leq k \leq N}$, $(t_N(0) \dots t_N^{(N)}(0))^t$ is Gaussian with covariance matrix AA^* .

$$\det(AA^*) = |\det A|^2 = \prod_{k=0}^N [k!^2 \binom{N}{k} |\phi'(0)|^{2k}] \neq 0 \quad (6.1)$$

because ϕ is a biholomorphism.

We again define $\kappa = 1 - \sqrt{\delta}$. Then if $\sup_{\partial D(0, \kappa)} |t_N| < 1$, for $0 \leq k \leq N$,

$$|t_N^{(k)}(0)| = \left| \frac{k!}{2\pi\sqrt{-1}} \int_{\partial D(0, \kappa)} \frac{t_N(u)}{u^{k+1}} du \right| \leq \frac{k!}{\kappa^k}.$$

Therefore,

$$\begin{aligned} \gamma_N(\sup_{\partial D(0, \kappa)} |t_N| < 1) &\leq \gamma_N\left\{(t_N(0), \dots, t_N^{(N)}(0)) \in \prod_{k=0}^N \bar{D}(0, \frac{k!}{\kappa^k})\right\} \\ &= \frac{1}{\pi^{N+1} \det(AA^*)} \int_{\prod_{k=0}^N \bar{D}(0, \frac{k!}{\kappa^k})} \exp\{-\eta^*(AA^*)^{-1}\eta\} d_{2(N+1)}\eta \\ &\leq \frac{\pi^{N+1} \prod_{k=0}^N \left(\frac{k!}{\kappa^k}\right)^2}{\pi^{N+1} \det(AA^*)}. \end{aligned}$$

By (6.1),

$$\begin{aligned}
\gamma_N(\sup_{\partial D(0,\kappa)} |t_N| < 1) &\leq \frac{\prod_{k=0}^N \left(\frac{k!}{\kappa^k}\right)^2}{\prod_{k=0}^N [k!^2 \binom{N}{k} |\phi'(0)|^{2k}]} \\
&= \left\{ \prod_{k=0}^N \left[\binom{N}{k} (\kappa |\phi'(0)|)^{2k} \right] \right\}^{-1} \\
&= \exp\{-Q_{\kappa|\phi'(0)|,1}(N)\} \\
&= \exp\{-(\log |\phi'(0)| + \log \kappa + \frac{1}{2})N^2 + o(N^2)\},
\end{aligned}$$

where the last equality is due to Lemma 2.2.

Similar as Lemma 4.9, we can show that if $t_N|_{\bar{D}(0,1)} \neq 0$, then outside an event of probability at most $e^{-e^N} + \exp\{-Q_{\kappa|\phi'(0)|,1}(N)\} = \exp\{-(\log |\phi'(0)| + \log \kappa + \frac{1}{2})N^2 + o(N^2)\}$,

$$\log \prod_{j=0}^N |t_N(z_j)| \leq \frac{o(N^2)}{\delta^2} + (N+1) \log |c_0|,$$

where $z_j = \kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}}$, $0 \leq j \leq N$.

$(t_N(z_0) \dots t_N(z_N))^t$ is complex Gaussian with covariance matrix

$$\begin{aligned}
\Sigma &= (\mathbb{E}_N(t_N(z_j) \overline{t_N(z_j)}))_{0 \leq i, j \leq N} = \left(\sum_{k=0}^N \binom{N}{k} (\phi(z_i))^k (\overline{\phi(z_j)})^k \right)_{0 \leq i, j \leq N} \\
&= \begin{bmatrix} \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_0) & \dots & \sqrt{\binom{N}{N}}(\phi(z_0))^N \\ \dots & \dots & \dots & \dots \\ \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_N) & \dots & \sqrt{\binom{N}{N}}(\phi(z_N))^N \end{bmatrix} \begin{bmatrix} \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_0) & \dots & \sqrt{\binom{N}{N}}(\phi(z_0))^N \\ \dots & \dots & \dots & \dots \\ \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_N) & \dots & \sqrt{\binom{N}{N}}(\phi(z_N))^N \end{bmatrix}^*
\end{aligned}$$

and

$$\begin{aligned}
\det \Sigma &= \prod_{k=0}^N \binom{N}{k} \prod_{0 \leq i < j \leq N} |\phi(z_i) - \phi(z_j)|^2, \\
\Rightarrow \log \det \Sigma &= \sum_{k=0}^N \log \binom{N}{k} + 2 \sum_{0 \leq i < j \leq N} \log |\phi(z_i) - \phi(z_j)|,
\end{aligned} \tag{6.2}$$

Next we will show that

$$2 \sum_{0 \leq i < j \leq N} \log |\phi(z_i) - \phi(z_j)| = N^2 \int_{\partial D(0,\kappa)} \int_{\partial D(0,\kappa)} \log |\phi(u_1) - \phi(u_2)| d\sigma_\kappa(u_1) d\sigma_\kappa(u_2) + o_\delta(N^2), \tag{6.3}$$

where $o_\delta(N^2)$ denotes a lower order term depending on δ .

Since

$$2 \sum_{0 \leq i < j \leq N} \log |\phi(z_i) - \phi(z_j)| = 2(N+1)^2 \sum_{0 \leq i < j \leq N} \frac{1}{(N+1)^2} \log |\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}}) - \phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})|$$

and

$$\begin{aligned}
&\int_{\partial D(0,\kappa)} \int_{\partial D(0,\kappa)} \log |\phi(u_1) - \phi(u_2)| d\sigma_\kappa(u_1) d\sigma_\kappa(u_2) \\
&= \int_0^1 \int_0^1 \log |\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})| dx dy \\
&= 2 \iint_{0 \leq x \leq y \leq 1} \log |\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})| dx dy,
\end{aligned}$$

it suffices to show that

$$\begin{aligned}
&\left| \sum_{0 \leq i < j \leq N} \frac{1}{(N+1)^2} \log |\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}}) - \phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})| - \iint_{0 \leq x \leq y \leq 1} \log |\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})| dx dy \right| \\
&= o_\delta(1).
\end{aligned}$$

Since ϕ is a biholomorphism in $D(0, 1)$, we set

$$\inf_{\bar{D}(0, \kappa)} |\phi'| = a(\delta) > 0.$$

And by Cauchy's inequality, we have

$$\sup_{\bar{D}(0, \kappa)} |\phi'| \leq O(\delta^{-1}).$$

For each N , denote

$$\Delta(N) = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i < j \leq N\},$$

the “far from diagonal” indices

$$FD(N) = \left\{ (i, j) \in \Delta(N) : \begin{array}{ll} \lfloor \sqrt{N+1} \rfloor + i \leq j \leq N - \lfloor \sqrt{N+1} \rfloor + i & \text{if } 0 \leq i \leq \lfloor \sqrt{N+1} \rfloor \\ \lfloor \sqrt{N+1} \rfloor + i \leq j \leq N & \text{if } \lfloor \sqrt{N+1} \rfloor < i \leq N - \lfloor \sqrt{N+1} \rfloor \\ j \in \emptyset & \text{if } i > N - \lfloor \sqrt{N+1} \rfloor \end{array} \right\},$$

$$\mathcal{FD}(N) = \bigcup_{(i, j) \in FD(N)} \left[\frac{i}{N+1}, \frac{i+1}{N+1} \right] \times \left[\frac{j}{N+1}, \frac{j+1}{N+1} \right],$$

and the “near diagonal” indices:

$$D(N) = \Delta(N) \setminus FD(N).$$

Then

$$|D(N)| = O(N^{\frac{3}{2}}),$$

and for $(i, j) \in FD(N)$,

$$\frac{i}{N+1} - \frac{j}{N+1} \geq (N+1)^{-\frac{1}{2}} \pmod{1}.$$

So

$$\begin{aligned} & \left| \sum_{0 \leq i < j \leq N} \frac{1}{(N+1)^2} \log |\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}}) - \phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})| - \iint_{0 \leq x \leq y \leq 1} \log |\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})| dx dy \right| \\ & \leq \sum_{(i, j) \in D(N)} \frac{1}{(N+1)^2} \left| \log |\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}}) - \phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})| \right| \\ & \quad + \sum_{(i, j) \in FD(N)} \int_{\frac{i}{N+1}}^{\frac{j+1}{N+1}} \int_{\frac{i}{N+1}}^{\frac{j+1}{N+1}} \left| \log |\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})| - \log |\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}}) - \phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})| \right| dx dy \\ & \quad + \left| \iint_{\mathcal{FD}(N)} \log |\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})| dx dy - \iint_{0 \leq x \leq y \leq 1} \log |\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})| dx dy \right| \\ & = I + II + III. \end{aligned}$$

$$\begin{aligned} & \frac{a(\delta)}{N+1} \leq |\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}}) - \phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})| \leq O(1) \quad \forall (i, j) \in D(N), \\ & \Rightarrow \left| \log |\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}}) - \phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})| \right| \leq |\log a(\delta)| + \log(N+1), \\ & \Rightarrow I \leq \frac{O(N^{\frac{3}{2}})}{N^2} [|\log a(\delta)| + \log(N+1)] = o_\delta(1). \end{aligned}$$

Since

$$\sup_{x-y \geq (N+1)^{-\frac{1}{2}} \pmod{1}} \|\nabla \log |\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})|\| \leq \frac{O(\delta^{-1})}{a(\delta)(N+1)^{-\frac{1}{2}}} = \frac{O(N^{\frac{1}{2}})}{\delta a(\delta)},$$

$$\begin{aligned} \Rightarrow II &\leq \frac{N^2}{(N+1)^2} \sup_{x-y \geq (N+1)^{-\frac{1}{2}} \pmod{1}} \|\nabla \log |\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})|\| O(N^{-1}) \\ &\leq \frac{O(N^{-\frac{1}{2}})}{\delta a(\delta)} = o_\delta(1). \end{aligned}$$

By a similar argument as Lemma 4.3, we have

$$\lim_{N \rightarrow \infty} \text{Vol}_{\mathbb{R}^2} (\mathcal{F} \mathcal{D}(N) \triangle \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq 1\}) = 0.$$

Furthermore, (6.4) and (6.5) below indicate that the function $\log |\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})|$ is L^1 over $[0, 1]^2$,

$$\Rightarrow III \leq o_\delta(1).$$

Thus, we have proved (6.3).

For $u_1, u_2 \in D(0, 1)$, define:

$$\psi(u_1, u_2) = \begin{cases} \frac{\phi(u_1) - \phi(u_2)}{u_1 - u_2} & \text{if } u_1 \neq u_2, \\ \phi'(u_1) & \text{if } u_1 = u_2. \end{cases}$$

Then ψ is continuous and nonzero in $D(0, 1) \times D(0, 1)$. Moreover, by removable singularity theorem, ψ is holomorphic in u_1 as well as u_2 . Therefore, $\log |\psi|$ is pluriharmonic in $D(0, 1) \times D(0, 1)$. By mean value equality,

$$\begin{aligned} &\int_{\partial D(0, \kappa)} \int_{\partial D(0, \kappa)} \log |\phi(u_1) - \phi(u_2)| d\sigma_\kappa(u_1) d\sigma_\kappa(u_2) \\ &= \int_{\partial D(0, \kappa)} \int_{\partial D(0, \kappa)} \log |\psi(u_1, u_2)| d\sigma_\kappa(u_1) d\sigma_\kappa(u_2) + \int_{\partial D(0, \kappa)} \int_{\partial D(0, \kappa)} \log |u_1 - u_2| d\sigma_\kappa(u_1) d\sigma_\kappa(u_2) \\ &= \log |\psi(0, 0)| + \log \kappa + \int_{\partial D(0, 1)} \int_{\partial D(0, 1)} \log |u_1 - u_2| d\sigma_1(u_1) d\sigma_1(u_2) \\ &= \log |\phi'(0)| + \log \kappa + \int_{\partial D(0, 1)} \int_{\partial D(0, 1)} \log |u_1 - u_2| d\sigma_1(u_1) d\sigma_1(u_2), \end{aligned} \tag{6.4}$$

$$\begin{aligned} &\int_{\partial D(0, 1)} \int_{\partial D(0, 1)} \log |u_1 - u_2| d\sigma_1(u_1) d\sigma_1(u_2) \\ &= \int_0^1 \int_0^1 \log |e^{2\pi\sqrt{-1}x} - e^{2\pi\sqrt{-1}y}| dx dy \\ &= \int_0^1 \log |1 - e^{2\pi\sqrt{-1}x}| dx \\ &= \int_{\partial D(0, 1)} \log |1 - z| d\sigma_1(z) \\ &= 0, \end{aligned} \tag{6.5}$$

where the last equality is due to Lebesgue's dominated convergence theorem.

(6.2)~(6.5) show that

$$\begin{aligned} \log \det \Sigma &= \sum_{k=0}^N \log \binom{N}{k} + (\log |\phi'(0)| + \log \kappa) N^2 + o_\delta(N^2) \\ &= (\log |\phi'(0)| + \log \kappa + \frac{1}{2}) N^2 + o_\delta(N^2). \end{aligned}$$

The remaining part is similar to Section 4. □

Remark 6.1. For $U = D(0, r)$, ϕ would be a rotation composed with a scaling by r . So $|\phi'(0)| = r$. Thus the upper bound in Theorem 0.5 is $-(\log r + \frac{1}{2})N^2 + o(N^2)$, which agrees with Corollary 0.4 in case of $r \geq 1$.

7. GENERALIZED HOLE PROBABILITIES OF $SU(2)$ POLYNOMIALS

If $n(r, N)$ denotes the number of zeros of $\tilde{s}_N(z)$ in $\bar{D}(0, r)$ counting multiplicity, then the hole probability $P_{0,1}(r, N)$ is just the first term of the sequence of the probabilities

$$P_{k,1}(r, N) = \gamma_N \{n(r, N) \leq k\}, \quad k \geq 0.$$

We call $P_{k,1}(r, N)$ a generalized hole probability because compared with the large degree or total number of zeros in \mathbb{C} of the polynomial \tilde{s}_N , any finite number k is negligible. It is a status of almost having no zero in $D(0, r)$. And by Theorem 0.6, it turns out that the generalized hole probabilities are numerically almost equal to the regular one.

Proof of Theorem 0.6. (4.20) implies that $\forall \eta > 0$,

$$\gamma_N \left\{ \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| \, d\sigma_r(u) > \frac{N}{2} \log(1+r^2) + \eta N \right\} \leq e^{-e^{\eta N}} \quad \text{for } N \gg 1. \quad (7.1)$$

We follow the notations in Section 5 except this time $m = 1$ and we take the number of partitions $p = 1$. The corresponding statement of Lemma 5.6 is

$$\gamma_N \left\{ \log \prod_{j=0}^{\lfloor \alpha_0 N \rfloor} |\zeta_j| > \frac{o(N^2)}{\delta^2} + (\lfloor \alpha_0 N \rfloor + 1) \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| \, d\sigma_r(u) \right\} \leq e^{-e^N} + e^{-R_{\kappa r,1}(N)},$$

where $\zeta_j = \tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}\frac{j}{\lfloor \alpha_0 N \rfloor + 1}})$, $0 \leq j \leq \lfloor \alpha_0 N \rfloor$. Here we do not need to assume $0 \notin \tilde{s}_N(\bar{D}(0, r))$ as in Lemma 5.6: the counterpart of II in (4.18) is

$$II = (\lfloor \alpha_0 N \rfloor + 1) \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| \int_H P_r(\kappa r e^{2\pi\sqrt{-1}x}, u) \, dx d\sigma_r(u).$$

Since $m = 1$ and $p = 1$, $H = [0, 1] \subset \mathbb{R}$,

$$\begin{aligned} II &= (\lfloor \alpha_0 N \rfloor + 1) \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| \int_0^1 P_r(\kappa r e^{2\pi\sqrt{-1}x}, u) \, dx d\sigma_r(u) \\ &= (\lfloor \alpha_0 N \rfloor + 1) \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| \, d\sigma_r(u). \end{aligned}$$

Therefore, $\forall \eta > 0$ small,

$$\begin{aligned} &\gamma_N \left\{ \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| \, d\sigma_r(u) \leq \frac{N}{2} \log(1+r^2) - \eta N \right\} \\ &\leq e^{-e^N} + e^{-R_{\kappa r,1}(N)} + \gamma_N \left\{ \prod_{j=0}^{\lfloor \alpha_0 N \rfloor} |\zeta_j| \leq \exp \left\{ \frac{o(N^2)}{\delta^2} + (\lfloor \alpha_0 N \rfloor + 1) \left[\frac{N}{2} \log(1+r^2) - \eta N \right] \right\} \right\}. \end{aligned} \quad (7.2)$$

Following the steps (4.28)~(4.30), we can show that

$$\begin{aligned} &\log \gamma_N \left\{ \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| \, d\sigma_r(u) \leq \frac{N}{2} \log(1+r^2) - \eta N \right\} \\ &\leq N(\lfloor \alpha_0 N \rfloor + 1)[\log(1+r^2) - 2\eta] - Q_{\kappa r,1,\alpha_0}(N) - 2\beta_1 \alpha_0^2 N^2 + \frac{o(N^2)}{\delta^2}. \end{aligned}$$

$$Q_{\kappa r,1,\alpha_0}(N) \sim N^2 \int_0^{\alpha_0} E_r(x) \, dx = \frac{1}{2} \alpha_0 [2 \log \kappa r + 1 - \log \alpha_0] N^2,$$

$$\begin{aligned} \beta_1 &= \int_0^1 x \log [2 \sin(\pi x)] \, dx \\ &= \int_0^1 (x - \frac{1}{2}) \log [2 \sin(\pi x)] \, dx + \frac{1}{2} \int_0^1 \log [2 \sin(\pi x)] \, dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} x \log [2 \sin \pi(x + \frac{1}{2})] \, dx + \frac{1}{2} \int_0^1 \log [2 \sin(\pi x)] \, dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} x \log [2 \cos(\pi x)] \, dx + \frac{1}{2} \int_0^1 \log [2 \sin(\pi x)] \, dx, \end{aligned}$$

as $\int_{-\frac{1}{2}}^0 x \log [2 \cos(\pi x)] dx$ and $\int_0^{\frac{1}{2}} x \log [2 \cos(\pi x)] dx$ both converge and $x \log [2 \cos(\pi x)]$ is odd,

$$\beta_1 = \frac{1}{2} \int_0^1 \log [2 \sin(\pi x)] dx = \frac{1}{2} \int_{\partial D(0,1)} \log |1-z| d\sigma_1(z),$$

which equals 0 as in (6.5). Thus

$$\begin{aligned} & \log \gamma_N \left\{ \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| d\sigma_r(u) \leq \frac{N}{2} \log(1+r^2) - \eta N \right\} \\ & \leq -\frac{1}{2} \alpha_0 [1 + 2 \log(\kappa r) - \log \alpha_0 - 2 \log(1+r^2) + 4\eta] N^2 + \frac{o(N^2)}{\delta^2}. \end{aligned} \quad (7.3)$$

On the other hand,

$$R_{\kappa r,1}(N) \sim N^2 \int_{E_{\kappa r}(x) \geq 0} E_{\kappa r}(x) dx. \quad (7.4)$$

Combine (7.2)~(7.4), and let $\delta \rightarrow 0+$, we get

$$\begin{aligned} & \log \gamma_N \left\{ \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| d\sigma_r(u) \leq \frac{N}{2} \log(1+r^2) - \eta N \right\} \\ & \leq -\min \left\{ \frac{1}{2} \alpha_0 [1 + 2 \log r - \log \alpha_0 - 2 \log(1+r^2) + 4\eta], \frac{1}{2} \alpha_0 [1 + 2 \log r - \log \alpha_0] \right\} N^2 + o(N^2) \\ & = -\frac{1}{2} \alpha_0 [1 + 2 \log r - \log \alpha_0 - 2 \log(1+r^2) + 4\eta] N^2 + o(N^2), \end{aligned} \quad (7.5)$$

for $0 < \eta < \frac{1}{2} \log(1+r^2)$. Since

$$\int_{E_r(x) \geq 0} E_r(x) dx = \frac{1}{2} \alpha_0 [1 + 2 \log r - \log \alpha_0] > 0 \Rightarrow 1 + 2 \log r - \log \alpha_0 > 0,$$

we can choose $0 < \eta < \frac{1}{2} \log(1+r^2)$ close to $\frac{1}{2} \log(1+r^2)$ such that

$$1 + 2 \log r - \log \alpha_0 - 2 \log(1+r^2) + 4\eta > 0.$$

Therefore (7.5) makes sense. Denote

$$F_\eta(r) = \frac{1}{2} \alpha_0 [1 + 2 \log r - \log \alpha_0 - 2 \log(1+r^2) + 4\eta],$$

so we have

$$\gamma_N \left\{ \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| d\sigma_r(u) \leq \frac{N}{2} \log(1+r^2) - \eta N \right\} \leq e^{-F_\eta(r) N^2 + o(N^2)}, \quad 0 < \eta < \frac{1}{2} \log(1+r^2). \quad (7.6)$$

Let $\rho > 1$ to be determined. By discarding a null set, we may assume $\tilde{s}_N(0) \neq 0$, $0 \notin \tilde{s}_N(\partial D(0,r))$ and $0 \notin \tilde{s}_N(\partial D(0, \rho^{-1}r))$.

So by Jensen's formula, almost surely,

$$\int_{\partial D(0,r)} \log |\tilde{s}_N(u)| d\sigma_r(u) = \log |c_0| + \int_0^r \frac{n(t, N)}{t} dt, \quad (7.7)$$

$$\int_{\partial D(0, \rho^{-1}r)} \log |\tilde{s}_N(u)| d\sigma_{\rho^{-1}r}(u) = \log |c_0| + \int_0^{\rho^{-1}r} \frac{n(t, N)}{t} dt. \quad (7.8)$$

Since $n(r, N)$ is increasing with respect to r ,

$$\begin{aligned} (7.7) \sim (7.8) & \Rightarrow \int_{\partial D(0,r)} \log |\tilde{s}_N(u)| d\sigma_r(u) - \int_{\partial D(0, \rho^{-1}r)} \log |\tilde{s}_N(u)| d\sigma_{\rho^{-1}r}(u) \\ & = \int_{\rho^{-1}r}^r \frac{n(t, N)}{t} dt \leq n(r, N) \log \rho, \\ & \Rightarrow n(r, N) \geq \frac{1}{\log \rho} \left[\int_{\partial D(0,r)} \log |\tilde{s}_N(u)| d\sigma_r(u) - \int_{\partial D(0, \rho^{-1}r)} \log |\tilde{s}_N(u)| d\sigma_{\rho^{-1}r}(u) \right]. \end{aligned} \quad (7.9)$$

(7.1) \Rightarrow For $\eta_1 > 0$, outside an event of probability at most $e^{-e^{\eta_1 N}}$,

$$\int_{\partial D(0, \rho^{-1}r)} \log |\tilde{s}_N(u)| d\sigma_{\rho^{-1}r}(u) \leq \frac{N}{2} \log(1 + \rho^{-2}r^2) + \eta_1 N, \quad (7.10)$$

(7.6) \Rightarrow For $0 < \eta_2 < \frac{1}{2} \log(1+r^2)$, outside an event of probability at most $e^{-F_{\eta_2}(r)N^2+o(N^2)}$,

$$\int_{\partial D(0,r)} \log |\tilde{s}_N(u)| d\sigma_r(u) \geq \frac{N}{2} \log(1+r^2) - \eta_2 N. \quad (7.11)$$

(7.9)~(7.11) \Rightarrow outside an event of probability at most $e^{-e^{\eta_1 N}} + e^{-F_{\eta_2}(r)N^2+o(N^2)}$,

$$n(r, N) \geq \frac{N}{\log \rho} \left[\frac{1}{2} \log(1+r^2) - \frac{1}{2} \log(1+\rho^{-2}r^2) - (\eta_1 + \eta_2) \right].$$

$$\Rightarrow \gamma_N \{n(r, N) < \frac{N}{\log \rho} \left[\frac{1}{2} \log(1+r^2) - \frac{1}{2} \log(1+\rho^{-2}r^2) - (\eta_1 + \eta_2) \right]\} \leq e^{-e^{\eta_1 N}} + e^{-F_{\eta_2}(r)N^2+o(N^2)},$$

where the right hand side is independent of ρ . We need to choose proper ρ , η_1 and η_2 .

$\forall \tau > 0$, we set

$$\frac{1}{\log \rho} \left[\frac{1}{2} \log(1+r^2) - \frac{1}{2} \log(1+\rho^{-2}r^2) - (\eta_1 + \eta_2) \right] = \tau,$$

$$\eta_1 + \eta_2 = \eta_\tau(\rho) := \frac{1}{2} \log(1+r^2) - \frac{1}{2} \log(1+\rho^{-2}r^2) - \tau \log \rho.$$

If $\tau > 0$ is small enough, $\rho_0(\tau) := \sqrt{\frac{1-\tau}{\tau}} r > 1$,

$$\eta'_\tau(\rho) = \frac{\rho^{-3}r^2}{1+\rho^{-2}r^2} - \frac{\tau}{\rho} = \frac{(1-\tau)r^2 - \tau\rho^2}{\rho(\rho^2+r^2)} \begin{cases} > 0 & \text{when } 1 < \rho < \rho_0, \\ = 0 & \text{when } \rho = \rho_0, \\ < 0 & \text{when } \rho > \rho_0. \end{cases}$$

$$\begin{aligned} \Rightarrow (\eta_1 + \eta_2)_{\max} &= \eta_\tau(\rho_0(\tau)) \\ &= \frac{1}{2} \log(1+r^2) - \frac{1}{2} \log\left(1 + \frac{\tau}{1-\tau}\right) - \tau \left[\frac{1}{2} \log(1-\tau) - \frac{1}{2} \log \tau + \log r \right] \\ &= \frac{1}{2} \log(1+r^2) + \frac{1}{2} \log(1-\tau) - \frac{\tau}{2} \log(1-\tau) + \frac{\tau}{2} \log \tau - \tau \log r \\ &= \frac{1}{2} \log(1+r^2) + \frac{1}{2} [\tau \log \tau + (1-\tau) \log(1-\tau) - 2\tau \log r]. \end{aligned}$$

For a fixed $r > 0$, we can choose smaller $\tau > 0$ if necessary so that

$$-\frac{1}{2} \log(1+r^2) < \tau \log \tau + (1-\tau) \log(1-\tau) - 2\tau \log r < 0.$$

This is possible since

$$\tau \log \tau + (1-\tau) \log(1-\tau) - 2\tau \log r < 0 \text{ if } 0 < \tau < \alpha_0$$

and

$$\lim_{\tau \rightarrow 0+} [\tau \log \tau + (1-\tau) \log(1-\tau) - 2\tau \log r] = 0.$$

Thus for such τ and the corresponding $\rho_0 = \rho_0(\tau)$,

$$\frac{1}{4} \log(1+r^2) < \eta_1 + \eta_2 = \eta_\tau(\rho_0) < \frac{1}{2} \log(1+r^2).$$

In this case, $\forall 0 < \eta_1 < \frac{1}{4} \log(1+r^2)$,

$$0 < \eta_2 = \frac{1}{2} \log(1+r^2) + \frac{1}{2} [\tau \log \tau + (1-\tau) \log(1-\tau) - 2\tau \log r] - \eta_1 < \frac{1}{2} \log(1+r^2),$$

$$\begin{aligned} \gamma_N \{n(r, N) < \tau N\} &= \gamma_N \left\{ n(r, N) < \frac{N}{\log \rho_0} \left[\frac{1}{2} \log(1+r^2) - \frac{1}{2} \log(1+\rho_0^{-2}r^2) - (\eta_1 + \eta_2) \right] \right\} \\ &\leq e^{-e^{\eta_1 N}} + e^{-F_{\eta_2}(r)N^2+o(N^2)}. \end{aligned}$$

$\forall k \geq 0$, for N large enough, $k < \tau N$,

$$\begin{aligned} \exp \left\{ -\frac{1}{2} \alpha_0 (1 + 2 \log r - \log \alpha_0) N^2 + o(N^2) \right\} &= P_{0,1}(r, N) \leq P_{k,1}(r, N) \leq \gamma_N \{n(r, N) < \tau N\} \\ &\leq e^{-e^{\eta_1 N}} + \exp \left\{ -\frac{1}{2} \alpha_0 \left\{ (1 + 2 \log r - \log \alpha_0) + 2 [\tau \log \tau + (1-\tau) \log(1-\tau) - 2\tau \log r] - 4\eta_1 \right\} N^2 + o(N^2) \right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} -\frac{1}{2}\alpha_0(1+2\log r - \log \alpha_0) &\leq \liminf_{N \rightarrow \infty} \frac{\log P_{k,1}(r, N)}{N^2} \leq \limsup_{N \rightarrow \infty} \frac{\log P_{k,1}(r, N)}{N^2} \\ &\leq -\frac{1}{2}\alpha_0 \left\{ (1+2\log r - \log \alpha_0) + 2[\tau \log \tau + (1-\tau) \log(1-\tau) - 2\tau \log r] - 4\eta_1 \right\}. \end{aligned}$$

Let $\eta_1 \rightarrow 0+$ and then $\tau \rightarrow 0+$,

$$\Rightarrow \lim_{N \rightarrow \infty} \frac{\log P_{k,1}(r, N)}{N^2} = -\frac{1}{2}\alpha_0(1+2\log r - \log \alpha_0) \Leftrightarrow \log P_{k,1}(r, N) \sim -\frac{1}{2}\alpha_0(1+2\log r - \log \alpha_0)N^2.$$

□

8. APPENDIX

We now prove the following lemma:

Lemma 8.1. *The coefficient of $g_{m,N}(\xi)$ in $\det W_{m,N}(\xi)$ equals 1.*

Proof. Let $\mathcal{S}_{m,N}$ be the set of bijections from $\Gamma_{m,N}$ to $\Lambda_{m,N}$ and $\forall \sigma \in \mathcal{S}_{m,N}$, $J \in \Gamma_{m,N}$, write $\sigma(J) = (\sigma_1(J), \dots, \sigma_m(J))$. Then

$$\det W_{m,N}(\xi) = \sum_{\sigma \in \mathcal{S}_{m,N}} \text{sgn}(\sigma) \prod_{J \in \Gamma_{m,N}} \xi_J^{\sigma(J)} = \sum_{\sigma \in \mathcal{S}_{m,N}} \text{sgn}(\sigma) \prod_{J \in \Gamma_{m,N}} \xi_{1,j_1}^{\sigma_1(J)} \dots \xi_{m,j_m}^{\sigma_m(J)}.$$

To find those $\sigma \in \mathcal{S}_{m,N}$ ending up with $g_{m,N}(\xi)$, it is equivalent to find σ satisfying $\forall 1 \leq i \leq m$,

$$\sum_{J \in \Gamma_{m,N}^{i,k}} \sigma_i(J) = \begin{cases} \binom{k+i-1}{i} \binom{N-k+m-i}{m-i} & 1 \leq k \leq N, \\ 0 & k = 0, \end{cases} \quad (8.1)$$

where the set $\Gamma_{m,N}^{i,k}$ is defined in (2.7). We are going to prove by induction that

$$\sigma(J) = (j_1, j_2 - j_1, \dots, j_m - j_{m-1}) \text{ for all } J \in \Gamma_{m,N}. \quad (8.2)$$

First of all, similar to $\Gamma_{m,N}^{i,k}$, we introduce

$$\Lambda_{m,N}^{i,k} = \{(k_1, \dots, k_m) \in \Lambda_{m,N} : k_1 + \dots + k_i = k\},$$

$$\Lambda_{m,N} = \bigsqcup_{k=0}^N \Lambda_{m,N}^{i,k}, \forall 1 \leq i \leq m \text{ and } |\Lambda_{m,N}^{i,k}| = \binom{k+i-1}{i-1} \binom{N-k+m-i}{m-i} = |\Gamma_{m,N}^{i,k}|.$$

When $i = 1$, (8.1) shows

$$\sum_{J \in \Gamma_{m,N}^{1,k}} \sigma_1(J) = k \binom{N-k+m-1}{m-1}, \quad 0 \leq k \leq N, \quad (8.3)$$

where the number of terms in the summation on the left is $|\Gamma_{m,N}^{1,k}| = \binom{N-k+m-1}{m-1} = |\Lambda_{m,N}^{1,k}|$, $\forall 0 \leq k \leq N$. Then

$$\begin{aligned} k = 0 \text{ in (8.3)} &\Rightarrow \sigma(\Gamma_{m,N}^{1,0}) = \Lambda_{m,N}^{1,0} \Rightarrow \sigma\left(\bigsqcup_{k=1}^N \Gamma_{m,N}^{1,k}\right) = \bigsqcup_{k=1}^N \Lambda_{m,N}^{1,k}, \\ k = 1 \text{ in (8.3)} &\Rightarrow \sigma(\Gamma_{m,N}^{1,1}) = \Lambda_{m,N}^{1,1} \Rightarrow \sigma\left(\bigsqcup_{k=2}^N \Gamma_{m,N}^{1,k}\right) = \bigsqcup_{k=2}^N \Lambda_{m,N}^{1,k}, \\ &\dots \\ k = N \text{ in (8.3)} &\Rightarrow \sigma(\Gamma_{m,N}^{1,N}) = \Lambda_{m,N}^{1,N}, \end{aligned}$$

$$\Rightarrow \sigma_1(J) = j_1, \quad \forall J \in \Gamma_{m,N}.$$

Now assume for some $1 \leq i \leq m-1$, $(\sigma_1 + \cdots + \sigma_i)(J) = j_i$, $\forall J \in \Gamma_{m,N}$. Then $\forall 1 \leq k \leq N$,

$$\begin{aligned} \sum_{J \in \Gamma_{m,N}^{i+1,k}} (\sigma_1 + \cdots + \sigma_{i+1})(J) &= \sum_{J \in \Gamma_{m,N}^{i+1,k}} [j_i + \sigma_{i+1}(J)] \\ &= \sum_{j=0}^k j |\Gamma_{m,N}^{i,j} \cap \Gamma_{m,N}^{i+1,k}| + \binom{k+i}{i+1} \binom{N-k+m-i-1}{m-i-1} \\ &= \sum_{j=0}^k j \binom{j+i-1}{i-1} \binom{N-k+m-i-1}{m-i-1} + \binom{k+i}{i+1} \binom{N-k+m-i-1}{m-i-1} \\ &= k \binom{k+i}{i} \binom{N-k+m-i-1}{m-i-1}, \end{aligned}$$

where the second term in the second equality comes from (8.1). And for $k=0$,

$$\sum_{J \in \Gamma_{m,N}^{i+1,0}} (\sigma_1 + \cdots + \sigma_{i+1})(J) = \sum_{J \in \Gamma_{m,N}^{i+1,0}} [j_i + \sigma_{i+1}(J)] = 0.$$

So $\forall 0 \leq k \leq N$,

$$\sum_{J \in \Gamma_{m,N}^{i+1,k}} (\sigma_1 + \cdots + \sigma_{i+1})(J) = k \binom{k+i}{i} \binom{N-k+m-i-1}{m-i-1}, \quad (8.4)$$

where the number of terms in the summation on the left is $|\Gamma_{m,N}^{i+1,k}| = \binom{k+i}{i} \binom{N-k+m-i-1}{m-i-1} = |\Lambda_{m,N}^{i+1,k}|$, $\forall 0 \leq k \leq N$.

$$\begin{aligned} k=0 \text{ in (8.4)} &\Rightarrow \sigma(\Gamma_{m,N}^{i+1,0}) = \Lambda_{m,N}^{i+1,0} \Rightarrow \sigma\left(\bigsqcup_{k=1}^N \Gamma_{m,N}^{i+1,k}\right) = \bigsqcup_{k=1}^N \Lambda_{m,N}^{i+1,k}, \\ k=1 \text{ in (8.4)} &\Rightarrow \sigma(\Gamma_{m,N}^{i+1,1}) = \Lambda_{m,N}^{i+1,1} \Rightarrow \sigma\left(\bigsqcup_{k=2}^N \Gamma_{m,N}^{i+1,k}\right) = \bigsqcup_{k=2}^N \Lambda_{m,N}^{i+1,k}, \\ &\dots \\ k=N \text{ in (8.4)} &\Rightarrow \sigma(\Gamma_{m,N}^{i+1,N}) = \Lambda_{m,N}^{i+1,N}, \\ &\Rightarrow (\sigma_1 + \cdots + \sigma_{i+1})(J) = j_{i+1}, \quad \forall J \in \Gamma_{m,N}. \end{aligned}$$

Thus, (8.2) is proved. And it is trivial to check that the σ defined in (8.2) satisfies all the equations in (8.1). This means that there is only one $\sigma \in \mathcal{S}_{m,N}$ that ends up with $g_{m,N}(\xi)$, and it turns out to be order preserving. Therefore,

$$\det W_{m,N}(\xi) = g_{m,N}(\xi) + \dots$$

□

REFERENCES

- [1] J.-P. Demailly, *Complex analytic and differential geometry* (2012), available at <http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf>.
- [2] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original. MR1288523 (95d:14001)
- [3] J. B. Hough, M. Krishnapur, Y. Peres, and B. Virág, *Zeros of Gaussian analytic functions and determinantal point processes*, University Lecture Series, vol. 51, American Mathematical Society, Providence, RI, 2009. MR2552864 (2011f:60090)
- [4] A. Nishry, *Asymptotics of the hole probability for zeros of random entire functions*, Int. Math. Res. Not. IMRN **15** (2010), 2925–2946, DOI 10.1093/imrn/rnp229. MR2673715 (2011f:60099)
- [5] Y. Peres and B. Virág, *Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process*, Acta Math. **194** (2005), no. 1, 1–35, DOI 10.1007/BF02392515. MR2231337 (2007m:60150)
- [6] M. Sodin, *Zeros of Gaussian analytic functions*, Math. Res. Lett. **7** (2000), no. 4, 371–381, DOI 10.4310/MRL.2000.v7.n4.a2. MR1783614 (2002d:32030)
- [7] M. Sodin and B. Tsirelson, *Random complex zeroes. III. Decay of the hole probability*, Israel J. Math. **147** (2005), 371–379, DOI 10.1007/BF02785373. MR2166369 (2007a:60028)
- [8] B. Shiffman and S. Zelditch, *Random polynomials with prescribed Newton polytope*, J. Amer. Math. Soc. **17** (2004), no. 1, 49–108 (electronic), DOI 10.1090/S0894-0347-03-00437-5. MR2015330 (2005e:60032)

- [9] B. Shiffman, S. Zelditch, and S. Zrebiec, *Overcrowding and hole probabilities for random zeros on complex manifolds*, Indiana Univ. Math. J. **57** (2008), no. 5, 1977–1997, DOI 10.1512/iumj.2008.57.3700. MR2463959 (2010b:32027)
- [10] S. Zrebiec, *The zeros of flat Gaussian random holomorphic functions on \mathbb{C}^n , and hole probability*, Michigan Math. J. **55** (2007), no. 2, 269–284, DOI 10.1307/mmj/1187646994. MR2369936 (2009e:60118)

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218, USA
E-mail address: jyzhu@math.jhu.edu